

On Form Factors in nested Bethe Ansatz systems

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March 2, 2013

Abstract

We investigate form factors of local operators in the multi-component Quantum Non-linear Schrödinger model, a prototype theory solvable by the so-called nested Bethe Ansatz. We determine the analytic properties of the infinite volume form factors using the coordinate Bethe Ansatz solution and we establish a connection with the finite volume matrix elements. In the two-component models we derive a set of recursion relations for the “magnonic form factors”, which are the matrix elements on the nested Bethe Ansatz states. In certain simple cases (involving states with only one spin-impurity) we obtain explicit solutions for the recursion relations.

1 Introduction

One of the goals of many-body quantum physics is the calculation of correlation functions of local observables. The form factor program is an approach to tackle this problem; it consists of the following three steps:

1. Finding the eigenstates of the system and inserting a complete set of states between the two (or more) local operators.
2. Evaluating the matrix elements (form factors) of the local operators.
3. Summing up the resulting spectral series.

In generic models these tasks present a fantastic challenge. However, the situation is quite different in one-dimensional integrable models, where there are exact methods available to compute the exact spectrum and the form factors.

One class of models where this program has been particularly successful is the Bethe Ansatz solvable theories related to the $sl(2)$ symmetric R -matrix, such as the Lieb-Liniger model (also known as the Quantum Nonlinear Schrödinger equation) [1, 2] and the XXX and XXZ spin chains [3, 4, 5, 6]. A very powerful approach is the Algebraic Bethe Ansatz, developed by the Leningrad-school [7, 8, 9], which led to important results concerning the scalar products of Bethe states [10, 11] and the form factors [12, 13]. Tremendous effort has been devoted to the calculation of correlation functions as well; we do not attempt here to review the literature, instead we refer the reader to the book [14] and the recent paper [15].

A different class of models are those non-relativistic theories, where the excitations over a fixed reference state have internal degrees of freedom, for example the multi-component Non-Linear Schrödinger equation, the $sl(N)$ symmetric spin chains, or the one-dimensional Hubbard model. The spectrum of these models can be obtained by the so-called nested Bethe Ansatz [16, 17, 18, 19, 20, 21, 22, 23], the algebraic formulation of which was worked out for the $sl(N)$ -related models in the papers [24, 25]. Although the nested Bethe Ansatz is successful in finding the spectrum, the construction of the eigenstates is rather complicated and there are far fewer results available than in the $sl(2)$ case. Norms of eigenstates were obtained in [26, 27, 28, 29, 30] and there are approaches to calculate the scalar products, see [31] and references therein. However, no compact and convenient formulas have been found yet, which would facilitate the computation of correlation functions.

In the non-relativistic models mentioned above there are explicit and exact representations known for the eigenstates of the system; this allows (at least in principle) for constructive methods to find the form factors and correlation functions. The situation is different in the realm of (massive) integrable relativistic QFT's [32]. These theories are typically investigated in infinite volume, the Hilbert-space is spanned by asymptotic scattering states defined using the Faddeev-Zamolodchikov algebra [33]. This construction does not allow the direct determination of form factors of local operators; an indirect method has been developed instead: the so-called form factor bootstrap program [34, 35, 36, 37, 38, 39, 40]. The idea is to establish the analytic properties of the form factors based on the requirement of locality, resulting in a closed set of equations also called “form factor axioms”. These equations are restrictive enough so that supplied with a few additional assumptions (possibly depending on the operator in question) they uniquely determine the form factors. In massive field theories it is sufficient to obtain explicit expressions for the form factors with a small number of particles, because they typically saturate the spectral series for the vacuum correlations even at small distances [41, 42].

One of the most important form factor axioms is the kinematic pole (or annihilation pole) property, which relates (N, M) form factors (matrix elements on an N -particle and an M -particle state) to $(N - 1, M - 1)$ form factors¹. It states that whenever particle rapidities from the two states approach each other the form factor has a simple pole (kinematic singularity) and the amplitude is given by the form factor with the corresponding particles not present and a pre-factor depending on the exact S -matrix of the theory [36, 37]. Similar singularity properties were also found in the models related to the $sl(2)$ -symmetric R -matrix in the framework of the Algebraic Bethe Ansatz. The pole structure of the scalar products of Bethe states was first established in [10], this led to the discovery of the celebrated Slavnov-formula [11], a determinant formula describing the scalar product of an eigenstate and an arbitrary Bethe state. These results refer to the finite volume states and they were derived using a quite general algebraic construction. Moreover, they were used to determine form factors of local operators, and in the case when both states are eigenstates, the singularity properties of the resulting form factors are found to be essentially the same as in the relativistic case. This has been noted recently in [43], where it was also shown that a special non-relativistic and small-coupling limit of the sinh-Gordon model form factors yields the known matrix elements of the Lieb-Liniger model.

We wish to note that form factors of local and composite non-local operators in the Lieb-Liniger model were also considered using the infinite volume Quantum Inverse Scattering Method [44, 45]. One result of this approach is the so-called quantum Rosales expansion, which expresses the local field operators using the non-local Faddeev-Zamolodchikov operators. The Rosales expansion can be used to read off explicit expressions for the form factors, and to establish their analytic properties [46], leading to the same kinematical pole equation (apart from the normalization) as the one found in [47, 14, 43].

The kinematical pole axiom also appears in the study of the form factors of the anti-ferromagnetic spin chain [48]. In this case the states involved are excitations above the infinite volume anti-ferromagnetic ground state, which is already filled with a finite density of elementary particles. In this respect the situation considered in [48] is different than in [10, 11, 43], where the states in question are elementary excitations over the reference state.

In this paper we contribute to the calculation of form factors in the multi-component Nonlinear Schrödinger equation. Inspired by the results of [43] we revisit the methods of relativistic QFT: we consider the analytic properties of form factors, set up recursion relations and make an attempt to find solutions to them, without trying to find manageable expressions for scalar products or related, more basic quantities.

In obtaining explicit solutions to the recursion relations we restrict ourselves to matrix elements on states with only a single spin-impurity. One of our motivations to investigate

¹In relativistic field theory the (N, M) form factors can be expressed in terms of the analytic continuation of the $(0, N + M)$ form factors using the so-called crossing relation. Then the kinematic pole is usually expressed in terms of the $(0, N)$ form factors relating them to the $(0, N - 2)$ matrix elements.

this subclass of form factors is provided by a recent experiment with ultracold atomic gas [49], where the motion of spin-impurities was studied in an otherwise polarized background. Our present results can form the basis for the theoretical investigation of such situations. Related questions were studied in a very recent article [50]. The paper [50] only considered the infinite coupling case, whereas our results for the form factors hold at arbitrary coupling strengths.

One of the main steps of the present work is the identification of the (un-normalized) form factors in finite and infinite volume. This result bears relevance also for integrable relativistic QFT, where related questions have been investigated recently [51, 52]. We give a few remarks on this issue in the Conclusions.

The structure of the paper is as follows. In section 2 we consider the one-component case (the Lieb-Liniger model) and establish a number of results about the form factors, using only the coordinate Bethe Ansatz wave functions. Although this section does not contain new results, it serves as a basis for the generalizations in later sections. In section 3 we recall the construction of the (infinite volume) Bethe Ansatz states in the multi-component case and we establish the properties of the form factors, in particular the kinematical pole property. Section 4 deals with the two-component case: the magnonic form factors are introduced, which are the matrix elements on the nested Bethe Ansatz states. A set of “magnonic form factor equations” is obtained. In section 5 we solve these equations in a number of simple cases, involving states with a single spin-impurity. The sections 3-5 are concerned with the infinite volume situation, the connection to the finite volume nested Bethe Ansatz states is made in section 6. Finally, section 7 is devoted to our conclusions.

2 The Lieb-Liniger model: Coordinate Bethe Ansatz

In this section we review the basic facts about the coordinate Bethe Ansatz solution of the one-component Bose gas, the Lieb-Liniger model. The ideas and results of this section will be the basis for our investigations of the multi-component systems in section 3, 4 and 6.

2.1 The model and the coordinate Bethe Ansatz solution

The second quantized form of the Hamiltonian is

$$H = \int_{-L/2}^{L/2} dx \left(\partial_x \Psi^\dagger \partial_x \Psi + c \Psi^\dagger \Psi^\dagger \Psi \Psi \right). \quad (2.1)$$

Here $\Psi(x, t)$ and $\Psi^\dagger(x, t)$ are canonical non-relativistic Bose fields satisfying

$$[\Psi(x, t), \Psi^\dagger(y, t)] = \delta(x - y). \quad (2.2)$$

We used the conventions $m = 1/2$ and $\hbar = 1$. The first quantized form of the Hamiltonian is

$$H = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j < l} \delta(x_j - x_l).$$

The parameter c is the coupling constant; in the present work we only consider the repulsive case $c > 0$.

In (2.1) L denotes the size of the system. We will consider both the infinite volume ($L = \infty$) and finite volume cases. In the latter case we always assume periodic boundary conditions.

The eigenstates of the Hamiltonian (2.1) can be constructed using the Bethe Ansatz [1, 2, 14]. The N -particle coordinate space wave function is given by

$$\chi_N(\{x\}_N | \{p\}_N) = \frac{1}{\sqrt{N!}} \sum_{\mathcal{P} \in S_N} (-1)^{[\mathcal{P}]} \exp \left\{ i \sum_j x_j (\mathcal{P}p)_j \right\} \prod_{j > k} \left((\mathcal{P}p)_j - (\mathcal{P}p)_k - ic\epsilon(x_j - x_k) \right), \quad (2.3)$$

where $\{p\}_N$ is the set of pseudo-momenta or rapidities, $\epsilon(x)$ is the sign function, and the $\mathcal{P} \in S_N$ are permutations. The total energy and momentum of the multi-particle state is

$$E_N = \sum_j p_j^2 \quad P_N = \sum_j p_j.$$

In the infinite volume case the wave function (2.3) is an eigenstate for arbitrary set of rapidities. Moreover, the Bethe states with real rapidities form a complete set of states [53, 54, 55]. Rapidities with non-zero imaginary parts are not allowed because they result in unbounded wave functions.

In the finite volume case periodic boundary conditions force the quasi-momenta to be solutions of the Bethe Ansatz equations:

$$e^{ip_j L} \prod_{k \neq j} \frac{p_j - p_k - ic}{p_j - p_k + ic} = 1, \quad j = 1 \dots N. \quad (2.4)$$

It is known that in the repulsive case ($c > 0$) considered here all solutions to the Bethe equations are given by real rapidities and they provide a complete set of states [56, 57].

In finite volume the wave function (2.3) is normalizable. The norm of the eigenstates is given by [58, 10]

$$\mathcal{N}^{LL}(\{p\}_N) = \int |\chi_N|^2 = \prod_{j < k} ((p_j - p_k)^2 + c^2) \times \det \mathcal{G}^{LL} \quad (2.5)$$

with

$$\mathcal{G}_{jk}^{LL} = \delta_{j,k} \left(L + \sum_{l=1}^N \varphi(p_j - p_l) \right) - \varphi(p_j - p_k) \quad (2.6)$$

and

$$\varphi(u) = \frac{2c}{u^2 + c^2}.$$

2.2 Form Factors in finite and infinite volume

We are interested in the form factors of the field and density operators:

$$\Psi(0) \quad \text{and} \quad \rho(0) = \Psi^\dagger(0)\Psi(0).$$

The finite volume form factors are defined as

$$\mathbb{F}_N^\Psi(\{p\}_N, \{k\}_{N+1}) = \frac{1}{\sqrt{N+1}} \int_{-L/2}^{L/2} dx_1 \dots dx_N \chi_N^*(x_1, \dots, x_N | \{p\}_N) \chi_{N+1}(0, x_1, \dots, x_N | \{k\}_{N+1}) \quad (2.7)$$

$$\mathbb{F}_N^\rho(\{p\}_N, \{k\}_N) = N \int_{-L/2}^{L/2} dx_1 \dots dx_{N-1} \chi_N^*(0, x_1, \dots, x_{N-1} | \{p\}_N) \chi_N(0, x_1, \dots, x_{N-1} | \{k\}_N). \quad (2.8)$$

As coordinate space integrals these form factors are well defined for arbitrary rapidities and they depend on the volume through the parameters

$$l(k_j) = e^{ik_j L} \quad l^*(p_j) = e^{-ip_j L}.$$

The dependence on these parameters was studied thoroughly using Algebraic Bethe Ansatz [59, 60, 47, 14]. In particular, it was shown that if the rapidities are solutions to the Bethe equations, and there are no coinciding rapidities, then the form factors do not depend on the volume explicitly (apart from possible overall phase factors).

Note that the form factors are defined using the un-normalized wave functions, therefore the actual finite volume matrix elements can be obtained as

$$\langle \{p\}_N | \Psi | \{k\}_{N+1} \rangle = \frac{\mathbb{F}_N^\Psi(\{p\}_N, \{k\}_{N+1})}{\sqrt{\mathcal{N}^{LL}(\{p\}_N) \mathcal{N}^{LL}(\{k\}_{N+1})}}, \quad (2.9)$$

and similarly for the density operator.

An alternative definition for the form factors can be given in infinite volume. In this case the real space integrals are not convergent due to the oscillating wave functions. However, they can be made convergent by introducing regulators $f_\varepsilon(x)$ in x -space. We choose

$$f_\varepsilon(x) \equiv f(\varepsilon|x|),$$

where $f(x) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a smooth function satisfying

$$\lim_{x \rightarrow 0} f(x) = 1 \quad \lim_{x \rightarrow \infty} f(x) = 0.$$

An example is given by $f(x) = e^{-x}$. It can be shown that for every $p \in \mathbb{R} \setminus \{0\}$

$$\lim_{\varepsilon \rightarrow 0} \int_0^\infty dx f_\varepsilon(x) e^{ipx} = \frac{i}{p} \quad \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^0 dx f_\varepsilon(x) e^{ipx} = -\frac{i}{p} \quad (2.10)$$

independently of the choice of $f(x)$. Actually (2.10) can be considered as a well-defined prescription to evaluate the infinite volume integrals.

Using this prescription the infinite volume form factors are defined as

$$\begin{aligned} \mathcal{F}_N^\Psi(\{p\}_N, \{k\}_{N+1}) &= \lim_{\varepsilon \rightarrow 0} \sqrt{N+1} \int_{-\infty}^\infty dx_1 \dots dx_N \prod_{j=1}^N f_\varepsilon(x_j) \\ &\quad \times \chi_N^*(x_1, \dots, x_N | p) \chi_{N+1}(0, x_1, \dots, x_N | k) \end{aligned} \quad (2.11)$$

$$\begin{aligned} \mathcal{F}_N^\rho(\{p\}_N, \{k\}_N) &= \lim_{\varepsilon \rightarrow 0} N \int_{-\infty}^\infty dx_1 \dots dx_{N-1} \prod_{j=1}^{N-1} f_\varepsilon(x_j) \\ &\quad \times \chi_N^*(0, x_1, \dots, x_{N-1} | p) \chi_N(0, x_1, \dots, x_{N-1} | k). \end{aligned} \quad (2.12)$$

The connection between the finite volume and infinite volume form factors is established by the following theorem.

Theorem 1. *The form factors are the same in finite and infinite volume. In other words, if both sets $\{p\}$ and $\{k\}$ are solution to the Bethe equations with a volume parameter L and there are no coinciding rapidities ($p_j \neq k_l$), then*

$$\mathbb{F}_N^\Psi(\{p\}_N, \{k\}_{N+1}) = \mathcal{F}_N^\Psi(\{p\}_N, \{k\}_{N+1}) \quad (2.13)$$

$$\mathbb{F}_N^\rho(\{p\}_N, \{k\}_N) = \mathcal{F}_N^\rho(\{p\}_N, \{k\}_N). \quad (2.14)$$

Proof. For simplicity we only consider the field operator and the case $N = 1$. Then we have to prove the equation

$$\lim_{\varepsilon \rightarrow 0} \int_{-\infty}^\infty dx f_\varepsilon(x) \chi_1^*(0|p) \chi_2(0, x|k_0, k_1) = \int_{-L/2}^{L/2} dx \chi_1^*(0|p) \chi_2(0, x|k_0, k_1). \quad (2.15)$$

The integrand consists of sums of free wave functions with certain amplitudes. Due to the insertion of the field operator the amplitudes depend on the sign of x , therefore the integrals have to be split into two parts:

$$\int_{-\infty}^\infty = \int_{-\infty}^0 + \int_0^\infty \quad \text{and} \quad \int_{-L/2}^{L/2} = \int_{-L/2}^0 + \int_0^{L/2}.$$

The integral of an exponential function can be evaluated in the infinite volume case by (2.10) whereas in the finite volume case it is given by the Newton-Leibniz formula. The essential step to prove (2.15) is to note that in the finite volume case those terms of the Newton-Leibniz formula which represent the contributions at $x = 0$ exactly coincide with the corresponding contributions of the infinite volume case. On the other hand, the contributions of the Newton-Leibniz formula corresponding to $-L/2$ and $L/2$ cancel each other owing to the periodicity of the Bethe wave function. Therefore, the two sides of (2.15) are indeed equal.

Similar arguments can be given in the case of higher particle form factors, and also for the form factors of the density operator. \square

It is clear from this derivation that the case of coinciding rapidities, in particular the problem of expectation values has to be treated separately. Whenever there are coinciding rapidities the infinite volume FF becomes divergent, the finite volume FF remains finite and can be expressed using the properly defined limits of the infinite volume ones [61]. In the framework of Algebraic Bethe Ansatz such relations were established for certain non-local operators related to correlation functions in [59, 60, 47], whereas local operators describing higher-body local correlations were considered recently in [62]. In this work we will only consider the case of non-coinciding rapidities.

2.3 Important properties of the form factors

The coordinate Bethe Ansatz wave functions (2.3) are completely anti-symmetric with respect to an exchange of two rapidities. Therefore

$$\begin{aligned}\mathcal{F}_N^\Psi(p_1, \dots, p_N | k_0, \dots, k_j, k_{j+1}, \dots, k_N) &= -\mathcal{F}_N^\Psi(p_1, \dots, p_N | k_0, \dots, k_{j+1}, k_j, \dots, k_N) \\ \mathcal{F}_N^\Psi(p_1, \dots, p_j, p_{j+1}, \dots, p_N | k_0, \dots, k_N) &= -\mathcal{F}_N^\Psi(p_1, \dots, p_{j+1}, p_j, \dots, p_N | k_0, \dots, k_N),\end{aligned}$$

and similarly for the density operator.

The form factors have kinematical pole singularities whenever $p_j \rightarrow k_l$ for some j, l . The residue of the pole is given by Theorem 2 below. However, before establishing the theorem we need the following lemma:

Lemma 1. *Let $D_j \subset \mathbb{R}^{N-1}$, $j = 0, \dots, N-1$ be the region*

$$x_1 < x_2 < \dots < x_j < 0 < x_{j+1} < \dots < x_{N-1}.$$

Then integrating over this region we obtain

$$\lim_{\varepsilon \rightarrow 0} \int_{D_j} e^{i \sum_{l=1}^{N-1} p_l x_l} \prod_{l=1}^{N-1} f_\varepsilon(x_l) = \prod_{l=1}^j \frac{-i}{\sum_{m=1}^l p_m} \prod_{l=j+1}^{N-1} \frac{i}{\sum_{m=l}^{N-1} p_m}. \quad (2.16)$$

Proof. The lemma is proven easily by induction, i.e. by performing first the integral over x_1 or x_{N-1} . \square

Theorem 2. *Let $p_N \rightarrow k_N$. Then the behaviour of \mathcal{F}_N^Ψ is given by*

$$\begin{aligned}\mathcal{F}_N^\Psi(p_1, \dots, p_N | k_0, \dots, k_N) &\sim \\ \frac{i}{k_N - p_N} &\left(\prod_{j=1}^{N-1} (p_{Nj} + ic) \prod_{j=1}^N (k_{Nj} - ic) - \prod_{j=1}^{N-1} (p_{Nj} - ic) \prod_{j=1}^N (k_{Nj} + ic) \right) \times \\ &\mathcal{F}_{N-1}^\Psi(p_1, \dots, p_{N-1} | k_0, \dots, k_{N-1}).\end{aligned} \quad (2.17)$$

Proof. The form factor is given by a sum over two sets of permutations and a sum over the different regions. The pole in $p_N - k_N$ only appears for those permutations when both rapidities are coupled to the same x_j . Moreover it follows from Lemma 1 that the singularity only appears in those regions, where either x_j is larger than all other coordinates (including $x = 0$ for the position of the field operator), or if x_j is smaller than all other coordinates. In

these cases the integral over x_j yields (assuming that the second largest or second smallest coordinate is x_l)

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \int_{x_l}^{\infty} dx_j f_{\varepsilon}(x_j) e^{i(k_N - p_N)x_j} &\rightarrow \frac{-1}{i(k_N - p_N)} e^{i(k_N - p_N)x_l} \sim \frac{-1}{i(k_N - p_N)} \\ \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{x_l} dx_j f_{\varepsilon}(x_j) e^{i(k_N - p_N)x_j} &\rightarrow \frac{1}{i(k_N - p_N)} e^{i(k_N - p_N)x_l} \sim \frac{1}{i(k_N - p_N)}. \end{aligned}$$

Collecting all these terms and adding the proper pre-factors which arise due to the rapidities p_N and k_N according to (2.3) we obtain (2.17). \square

There is an analogous relation for the density operator. The form factors of both operators have the structure

$$\mathcal{F}_N(\{p\}, \{k\}) = \prod_{j < l} (k_j - k_l) \prod_{j > l} (p_l - p_j) \prod_{j, l} \frac{1}{p_j - k_l} \times P_N(\{p\}|\{k\}), \quad (2.18)$$

where $P_N(\{p\}|\{k\})$ are polynomials symmetric in both sets of variables. The degrees of the polynomials in their variables are established by the following lemmas.

Lemma 2. *The asymptotic behaviour of the field operator form factor is p_1^{N-3} at $p_1 \rightarrow \infty$ and k_0^{N-1} at $k_0 \rightarrow \infty$.*

Proof. At $p_1 \rightarrow \infty$ the overall degree of the wave function is p_1^{N-1} . The leading term factorizes: the amplitude does not depend on the position of the particle with rapidity p_1 . Therefore the integral over the coordinate attached to p_1 can be performed over the whole real line and the regularization scheme yields

$$\lim_{\varepsilon \rightarrow 0} \int_{\mathbb{R}} f_{\varepsilon}(x_1) e^{ip_1 x_1} = \lim_{\varepsilon \rightarrow 0} \left(\int_{\mathbb{R}^+} f_{\varepsilon}(x_1) e^{ip_1 x_1} + \int_{\mathbb{R}^-} f_{\varepsilon}(x_1) e^{ip_1 x_1} \right) = \frac{i}{p_1} + \frac{-i}{p_1} = 0.$$

Therefore the overall degree of the form factor is determined by the sub-leading terms of order p_1^{N-2} . The highest possible degree of the coordinate space integrals is p_1^{-1} , resulting in an overall degree of p_1^{N-3} .

At $k_0 \rightarrow \infty$ the leading term in the real space integral for the form factor is given by those terms where k_0 is attached to the coordinate $x_0 = 0$. To leading order the remaining wave function is proportional to a Bethe Ansatz state with the rapidities $\{k_1, \dots, k_N\}$ and the integrals yield the scalar product $\langle p_1, \dots, p_N | k_1, \dots, k_N \rangle$. It is assumed that the rapidities are different therefore this scalar product vanishes. The sub-leading terms in the wave function then yield an overall degree of k_0^{N-1} . \square

It follows from the Lemma above and the factorization (2.18) that in the case of the field operator P_N is of degree $N - 1$ in all of the p_j and the k_j variables. Therefore the recursion relations (2.17) contain enough information which completely determine the form factors; the polynomial P_N can be reconstructed using the Lagrange interpolation procedure [47].

In the case of the density operator the polynomial $P_N(\{p\}|\{k\})$, which is symmetric with respect to the exchange $\{p\} \leftrightarrow \{k\}$, has the following asymptotic behaviour:

Lemma 3. *The density form factor satisfies the asymptotics*

$$\lim_{p_1 \rightarrow \infty} \frac{F_N^p(p_1 \dots p_N | k_1 \dots k_N)}{p_1^{N-1}} = (-1)^{N-1} F_{N-1}^{\Psi}(p_2 \dots p_N | k_1 \dots k_N). \quad (2.19)$$

Proof. The leading terms in the $p_1 \rightarrow \infty$ limit are given by those permutations where p_1 is attached to the coordinate 0, in other words it is not integrated over. Concerning these terms the following relation can be read off from (2.3)

$$\lim_{p_1 \rightarrow \infty} \frac{\chi_N(0, x_1, \dots, x_{N-1} | p_1, \dots, p_N)}{p_1^{N-1}} \sim \frac{(-1)^{N-1}}{\sqrt{N}} \chi_{N-1}(x_1, \dots, x_{N-1} | p_2, \dots, p_N), \quad (2.20)$$

where the sign \sim indicates that on the l.h.s. only those permutations $\mathcal{P} \in S_N$ are kept which leave p_1 at the first place. The statement of the theorem then follows directly from (2.20) and the definitions (2.11)-(2.12). \square

It follows that P_N is of order N in all of its variables. The kinematic recursion relations together with the condition (2.19) completely fix the form factor.

2.4 Determinant formulas for the form factors

For the sake of completeness we present here explicit determinant formulas for the form factors, which will be the basis for the generalizations in section 5.

In the case of the field operator the form factor can be expressed as [13, 63]

$$\mathcal{F}_N^\Psi = \prod_{j \neq l} ((k_j - k_l)^2 + c^2) \det(S_{jl} - S_{N+1,l}).$$

Here S is an $N \times N$ matrix defined as

$$S_{jl} = t(p_j, k_l) \frac{\prod_{m=1}^N (p_m - k_j + ic)}{\prod_{m=0}^N (k_m - k_j + ic)} - t(k_l, p_j) \frac{\prod_{m=1}^N (k_j - p_m + ic)}{\prod_{m=0}^N (k_j - k_m + ic)},$$

with

$$t(u) = \frac{-c}{u(u+ic)}$$

The first three examples are given explicitly by

$$\begin{aligned} \mathcal{F}_0^\Psi(k) &= 1, & \mathcal{F}_1^\Psi(p|k_0, k_1) &= \frac{2c(k_0 - k_1)}{(k_0 - p)(k_1 - p)}, \\ \mathcal{F}_2^\Psi(p_1, p_2|k_0, k_1, k_2) &= \frac{-4c^2(k_0 - k_1)(k_1 - k_2)(k_0 - k_2)(p_1 - p_2)}{(k_0 - p_1)(k_0 - p_2)(k_1 - p_1)(k_1 - p_2)(k_2 - p_1)(k_2 - p_2)} \times \\ &\times (c^2 - k_0 k_1 - k_0 k_2 - k_1 k_2 + (k_0 + k_1 + k_2)(p_1 + p_2) - 3p_1 p_2). \end{aligned}$$

Concerning the density operator, the first determinant formula was established in [12]. In the present work we will use an independent representation, which is easily derived from the results of [64]:²

$$\mathcal{F}_N^\rho(\{p\}, \{k\}) = \frac{-i}{c} (-1)^{N(N+1)/2} \prod_{o=1}^N \prod_{l=1}^N (k_o - p_l + ic) \times \det V. \quad (2.21)$$

Here V is an $(N+1) \times (N+1)$ matrix with entries

$$\begin{aligned} V_{jl} &= \tilde{t}(k_j, p_l) + \tilde{t}(p_l, k_j) \prod_{o=1}^N \frac{(p_l - k_o + ic)(p_l - p_o - ic)}{(p_l - k_o - ic)(p_l - p_o + ic)} & j, l = 1 \dots N \\ V_{N+1,j} &= \prod_{o=1}^N \frac{p_o - p_j + ic}{k_o - p_j + ic} & \text{and } V_{j,N+1} = 1, & j = 1 \dots N \end{aligned} \quad (2.22)$$

$$V_{N+1,N+1} = 0,$$

and

$$\tilde{t}(u) = \frac{-i}{u(u+ic)}.$$

We wish to note that (2.21) can be written alternatively as an $N \times N$ determinant, but it is useful to keep this form, which makes it possible to find generalizations in section 5.

²The final form of the determinant formula (2.21) (and the generalizations (5.14) and (5.19)) was suggested by Jean-Sébastien Caux. The result given in [64] was expressed as a sum of N determinants, whereas the present formula is given by a single determinant, making it more convenient for numerical calculations.

The first two cases are given explicitly as

$$\mathcal{F}_1^\rho(p|k) = 1$$

$$\mathcal{F}_2^\rho(p_1, p_2|k_1, k_2) = \frac{-2c(k_1 + k_2 - p_1 - p_2)^2(k_1 - k_2)(p_1 - p_2)}{(k_1 - p_1)(k_1 - p_2)(k_2 - p_1)(k_2 - p_2)}.$$

2.5 An alternative representation for the form factors

The determinant formulas of the previous subsection are very convenient for both analytical and numerical analysis of the correlation functions. However, it is possible to find alternative representations, which might not seem as useful at first sight, but which might give clues for the calculation of form factors in nested Bethe Ansatz systems.

One such representation can be derived using results of the form factor bootstrap program of relativistic integrable QFT's [38]. The papers [65, 66] considered the form factors of breathers (in particular the lowest-lying breathers) in the sine-Gordon model, and they arrived at formulas, which give the form factors with a total number of N particles as a sum of 2^N terms. From this result it is possible to derive formulas for the Lieb-Liniger model, first performing an analytic continuation in the coupling constant to get the form factors of the sinh-Gordon model, and then using a non-relativistic and small-coupling limit as explained in [67, 68, 43].

In the case of the field operator we obtain the formula

$$\mathcal{F}_N^\Psi(\{p\}_N|\{k\}_{N+1}) = \frac{P}{2c^N} \prod_{j,k} \frac{1}{k_j - p_l}. \quad (2.23)$$

Here the polynomial P is given by

$$\begin{aligned} P = & \sum_{\alpha_j=0,1} \sum_{\beta_l=0,1} (-1)^{\sum_j \alpha_j + \sum_l \beta_l} \prod_{i1 < i2} (p_{i1} - p_{i2} + (\alpha_{i1} - \alpha_{i2})ic) \times \\ & \times \prod_{i1 < i2} (k_{i1} - k_{i2} + (\beta_{i1} - \beta_{i2})ic) \prod_{i1, i2} (p_{i1} - k_{i2} - (\alpha_{i1} - \beta_{i2})ic) \times \\ & \times \left(\sum_{j=1}^N (-1)^{\alpha_j} + \sum_{j=1}^{N+1} (-1)^{\beta_j} \right). \end{aligned} \quad (2.24)$$

The summation is performed over the variables $\alpha_j = 0, 1$ with $j = 1 \dots N$ and $\beta_j = 0, 1$, with $j = 0 \dots N$. In general the number of the α_j and β_j variables coincides with the number of particles in the bra and ket states, respectively.

In the case of the density operator the corresponding formula reads

$$\mathcal{F}_N^\rho(\{p\}_N|\{k\}_N, \mu) = \frac{-P}{8c^{N-1}} \prod_{j,k} \frac{1}{k_j - p_l} \quad (2.25)$$

with

$$\begin{aligned} P = & \sum_{\alpha_j=0,1} \sum_{\beta_l=0,1} (-1)^{\sum_j \alpha_j + \sum_l \beta_l} \prod_{i1 < i2} (p_{i1} - p_{i2} + (\alpha_{i1} - \alpha_{i2})ic) \times \\ & \times \prod_{i1 < i2} (k_{i1} - k_{i2} + (\beta_{i1} - \beta_{i2})ic) \prod_{i1, i2} (p_{i1} - k_{i2} - (\alpha_{i1} - \beta_{i2})ic) \times \\ & \times \left(\sum_{j=1}^N ((-1)^{\alpha_j} + (-1)^{\beta_j}) \right)^2. \end{aligned} \quad (2.26)$$

Note that the only difference between formulas (2.24) and (2.26) is the last factor, which is called the “p-function” in the original papers [65, 66].

It can be shown that formulas (2.23)-(2.25) satisfy all necessary conditions established in subsection 2.3, therefore they describe the field operator and density form factors indeed. In the case of (2.25) the factorization condition (2.19) is also easily checked using

$$\left(1 + \sum_{j=2}^N (-1)^{\alpha_j} + \sum_{j=1}^N (-1)^{\beta_j}\right)^2 - \left(-1 + \sum_{j=2}^N (-1)^{\alpha_j} + \sum_{j=1}^N (-1)^{\beta_j}\right)^2 = 4 \left(\sum_{j=2}^N (-1)^{\alpha_j} + \sum_{j=1}^N (-1)^{\beta_j}\right).$$

3 Multi-component systems: Coordinate Bethe Ansatz

In this section we consider the general K -component models in infinite volume. In second quantized form the Hamiltonian is

$$H = \int_{-\infty}^{\infty} dx \left(\partial_x \Psi_j^\dagger \partial_x \Psi_j + c \Psi_l^\dagger \Psi_j^\dagger \Psi_j \Psi_l \right). \quad (3.1)$$

Here $\Psi_j(x, t)$ and $\Psi_j^\dagger(x, t)$, $j = 1 \dots K$ are canonical non-relativistic Bose or Fermi fields satisfying

$$\Psi_j(x, t) \Psi_l^\dagger(y, t) - \sigma \Psi_l^\dagger(y, t) \Psi_j(x, t) = \delta_{jl} \delta(x - y), \quad (3.2)$$

where $\sigma = 1$ for bosons ($\sigma = -1$ for fermions), respectively.

In first quantized form the Hamiltonian is

$$H = - \sum_{j=1}^N \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j < l} \delta(x_j - x_l)$$

in both cases. Note that the above Hamiltonian is completely spin-independent; the interaction between the different spin components arises as the effect of the statistics of the wave function.

The construction of the eigenstates of the Hamiltonian (3.1) was established in the papers [16, 17, 18, 19, 20] (for a more general scheme see [69]). In the following we collect the main results of this procedure; our focus will be on the form factors and their analytic properties.

3.1 The wave functions

Before constructing the coordinate Bethe Ansatz wave functions we need to introduce a few basic objects and notations.

Consider the vector space $V = \mathbb{C}^K$. Consider also the N -fold tensor product

$$V^{(N)} = \otimes^N V$$

and a representation ρ of the permutation group S_N on $V^{(N)}$. Let ρ_{ab} denote the action corresponding to the elementary exchange (ab) . In the physical cases

$$\rho_{ab} = \sigma P_{ab},$$

where P_{ab} is the permutation operator between the vector spaces V_a and V_b and $\sigma = 1$ ($\sigma = -1$) in the bosonic (fermionic) case, respectively.

Consider also a set of parameters $\{p\}_N$, which will play the role of particle rapidities for the Bethe wave function.

We introduce the operators [19, 69]

$$Y_{jk}^{ab} = \frac{(p_j - p_k) \rho_{ab} - ic}{p_j - p_k + ic}. \quad (3.3)$$

Here it is understood that Y_{jk}^{ab} acts only on the vector spaces a and b and the indices jk stand for the rapidities entering the expression (3.3).

The operators (3.3) satisfy the unitarity condition and the Yang-Baxter equations:

$$Y_{jk}^{ab} Y_{kj}^{ab} = 1 \quad (3.4)$$

$$Y_{jk}^{ab} Y_{ik}^{bc} Y_{ij}^{ab} = Y_{ij}^{bc} Y_{ik}^{ab} Y_{jk}^{bc}. \quad (3.5)$$

In the following we attach the rapidities to the vector spaces. To every permutation of the rapidities $Q \in S_N$ we associate a configuration

$$V_1^{(Qp)_1} \otimes V_2^{(Qp)_2} \otimes \dots \otimes V_N^{(Qp)_N}. \quad (3.6)$$

We define an operator $\mathcal{Q}(Q, \{p\}) : (S_N \times \mathbb{C}^N) \rightarrow \text{End}(V^{(N)})$ as follows. The permutation $Q \in S_N$ is re-constructed as a product of elementary permutations and to every exchange of rapidities

$$V_a^{p_j} \otimes V_b^{p_k} \rightarrow V_a^{p_k} \otimes V_b^{p_j} \quad (3.7)$$

we associate the action of Y_{jk}^{ab} ; the operator $\mathcal{Q}(Q, \{p\})$ is defined as the product of the Y_{jk}^{ab} matrices. This definition leads to the property

$$\mathcal{Q}(Q_2, Q_1 p) \mathcal{Q}(Q_1, p) = \mathcal{Q}(Q_2 Q_1, p). \quad (3.8)$$

The consistency of the construction is guaranteed by the Yang-Baxter equation (3.5).

Now we are in a position to construct the vector valued wave functions:

$$\chi_N : \mathbb{R}^N \rightarrow V^{(N)}.$$

The wave functions depend on the (ordered) set of rapidities $\{p\}_N$ and an arbitrary (fixed) vector $\omega_N \in V^{(N)}$, which is a parameter describing the polarization of the wave function. It will be specified in the two-component case in section 4.

We define the fundamental domain as

$$x_j > x_l \quad \text{iff} \quad j > l. \quad (3.9)$$

In this region the wave function is

$$\chi_N(x|p, \omega_N) = \frac{1}{\sqrt{N!}} \sum_{Q \in S_N} e^{i\langle Qp|x \rangle} \mathcal{Q}(Q, p) \omega_N. \quad (3.10)$$

It can be extended to \mathbb{R}^N by symmetry. To write down the formal relation we need the representation $\rho(Q)$ of the permutation Q which is such that

$$(Qx)_1 < \dots < (Qx)_N.$$

Then the wave function in \mathbb{R}^N reads

$$\chi_N(x|p, \omega_N) = \frac{1}{\sqrt{N!}} \rho(Q^{-1}) \sum_{R \in S_N} e^{i\langle Rp|Qx \rangle} \mathcal{Q}(R, p) \omega_N. \quad (3.11)$$

It can be shown that the wave function defined this way is an eigenstate of the Hamiltonian (3.1) for arbitrary $\{p\}_N$ and ω_N . The total energy and the momentum is given by

$$E_N = \sum_j p_j^2 \quad P_N = \sum_j p_j,$$

the vector ω_N only determines the polarization of the wave function.

We wish to note that the matrix

$$X_{jk}^{ab} = P_{ab} Y_{jk}^{ab} = \frac{\sigma(p_j - p_k) - ic P_{ab}}{p_j - p_k + ic} \quad (3.12)$$

can be interpreted as the two-particle S-matrix of the theory. Therefore, the individual coefficients in (3.11) describe the two-particle scattering events.

It is important to establish the exchange property of the wave function with respect to an exchange of rapidities:

Lemma 4.

$$\chi_N(x|p_1, \dots, p_j, p_{j+1}, \dots, p_N, \omega_N) = \chi_N(x|p_1, \dots, p_{j+1}, p_j, \dots, p_N, Y_{j,j+1}^{j,j+1} \omega_N). \quad (3.13)$$

Proof. It is enough to consider the fundamental region. Let us denote the exchange ($j \leftrightarrow j+1$) by P_j . We introduce a new summation variable $Q' = QP_j$ leading to

$$\chi_N(x|p_1, \dots, p_j, p_{j+1}, \dots, p_N, \omega_N) = \frac{1}{\sqrt{N!}} \sum_{Q' \in S_N} e^{i\langle Q' P_j p | x \rangle} \mathcal{Q}(Q' P_j, p) \omega_N.$$

It follows from (3.8) that

$$\mathcal{Q}(Q' P_j, p) = \mathcal{Q}(Q', P_j p) \mathcal{Q}(P_j, p) = \mathcal{Q}(Q', P_j p) Y_{j,j+1}^{j,j+1}.$$

Therefore

$$\begin{aligned} \chi_N(x|p_1, \dots, p_j, p_{j+1}, \dots, p_N, \omega_N) &= \frac{1}{\sqrt{N!}} \sum_{Q' \in S_N} e^{i\langle Q' P_j p | x \rangle} \mathcal{Q}(Q', P_j p) Y_{j,j+1}^{j,j+1} \omega_N = \\ &= \chi_N(x|p_1, \dots, p_{j+1}, p_j, \dots, p_N, Y_{j,j+1}^{j,j+1} \omega_N). \end{aligned}$$

□

3.2 The Form Factors

We are interested in the form factors of the field operators $\Psi_l(0)$ and the bilinear operators $\rho_{jl}(0) = \Psi_j^\dagger(0) \Psi_l(0)$. The latter encompass the density operators of particles with a given spin (when $j = l$) and also the spin-flip operators (when $j \neq l$).

The (infinite volume) form factors are co-vector valued functions of the rapidities. Evaluated on two vectors $\psi_N \in V^{(N)}$ and $\phi_{N+1} \in V^{(N+1)}$ they are given as the coordinate space integrals

$$\begin{aligned} \mathcal{F}_N^l(\{p\}_N, \{k\}_{N+1})(\psi_N, \phi_{N+1}) &= \lim_{\varepsilon \rightarrow 0} \sqrt{N+1} \int_{-\infty}^{\infty} dx_1 \dots dx_N \prod_{j=1}^N f_\varepsilon(x_j) \times \\ &\quad \left\langle \chi_N(x_1, \dots, x_N | \{p\}_N, \psi_N) \middle| U_l^{(0)} \chi_{N+1}(0, x_1, \dots, x_N | \{k\}_{N+1}, \phi_{N+1}) \right\rangle_N \end{aligned} \quad (3.14)$$

$$\begin{aligned} \mathcal{F}_N^{jl}(\{p\}_N, \{k\}_N)(\psi_N, \phi_{N+1}) &= \lim_{\varepsilon \rightarrow 0} N \int_{-\infty}^{\infty} dx_1 \dots dx_{N-1} \prod_{j=1}^{N-1} f_\varepsilon(x_j) \times \\ &\quad \left\langle U_j^{(1)} \chi_N(p | 0, x_1, \dots, x_{N-1}, \psi_N) \middle| U_l^{(1)} \chi_N(k | 0, x_1, \dots, x_{N-1}, \phi_N) \right\rangle_{N-1}. \end{aligned} \quad (3.15)$$

Here $U_l^{(j)}$ is an operator

$$U_l^{(j)} : V^{(N)} \rightarrow V^{(N-1)} \quad (3.16)$$

which acts by taking the scalar product with the unity vector e_l in the j -th vector space and leaving the others invariant:

$$\begin{aligned} U_l^{(j)}(e_{a_1} \otimes e_{a_2} \otimes \dots \otimes e_{a_{j-1}} \otimes e_{a_j} \otimes e_{a_{j+1}} \otimes \dots \otimes e_{a_N}) &= \\ &= \delta_{l, a_j} (e_{a_1} \otimes e_{a_2} \otimes \dots \otimes e_{a_{j-1}} \otimes e_{a_{j+1}} \otimes \dots \otimes e_{a_N}) \end{aligned}$$

The scalar products in (3.14)-(3.15) are the canonical ones in $V^{(N)}$ and $V^{(N-1)}$, respectively. In order to conform with our previous notations, in the case of the field operator the indexation of the vector spaces corresponding to the rapidities $\{k\}_{N+1}$ is given by

$$V^{(N+1)} = V_0^{k_0} \otimes V_1^{k_1} \otimes \dots \otimes V_N^{k_N}. \quad (3.17)$$

In the following we establish the analytic properties of the form factors. The behaviour under the exchange of rapidities follows simply from the properties (3.13) of the Bethe Ansatz wave functions:

$$\begin{aligned}\mathcal{F}_N^l(p_1, \dots, p_N | k_0, \dots, k_j, k_{j+1}, \dots, k_N)(\phi_N, \psi_{N+1}) &= \\ F_N^l(p_1, \dots, p_N | k_0, \dots, k_{j+1}, k_j, \dots, k_N)(\phi_N, \hat{S}_{j,j+1} \psi_{N+1}) & \\ \mathcal{F}_N^l(p_1, \dots, p_j, p_{j+1}, \dots, p_N | k_0, \dots, k_N)(\phi_N, \psi_{N+1}) &= \\ F_N^l(p_1, \dots, p_{j+1}, p_j, \dots, p_N | k_0, \dots, k_N)(\hat{S}^{j,j+1} \phi_N, \psi_{N+1}). &\end{aligned}\quad (3.18)$$

Here we introduced the short-hand notation

$$\hat{S}_{j,l} = Y_{jl}^{jl}(\{k\}) = \frac{(k_j - k_l)\rho_{jl} - ic}{k_j - k_l + ic} \quad \hat{S}^{j,l} = Y_{jl}^{jl}(\{p\}) = \frac{(p_j - p_l)\rho_{jl} - ic}{p_j - p_l + ic}. \quad (3.19)$$

Analogous relations hold for the bilinear operators as well.

The singularity properties of the form factors are more involved. All the poles of the form factors arise from the coordinate space integrals and therefore the positions of the poles are identical with those in the scalar case, ie. the form factors have poles at $p_j = k_l$, whenever two rapidities from the two sides coincide. To write down the residues we introduce the following notation:

$$Id_{1,0} \otimes \mathcal{F}_{N-1} \quad \text{and} \quad \mathcal{F}_{N-1} \otimes Id_{N,N}$$

are operations where it is understood that a trace is taken with respect to the corresponding spaces of ψ_N and ϕ_{N+1} and the remaining vector spaces are substituted in the form factor. For example

$$\begin{aligned}(Id_{1,0} \otimes \mathcal{F}_{N-1})(e_{a_1} \otimes \dots \otimes e_{a_N}, e_{b_0} \otimes \dots \otimes e_{b_N}) &= \\ \delta_{a_1, b_0} \mathcal{F}_{N-1}(e_{a_2} \otimes \dots \otimes e_{a_N}, e_{b_1} \otimes \dots \otimes e_{b_N}) &\end{aligned}$$

With this notation the kinematical poles are given by the following theorem.

Theorem 3. *The pole of the field operator form factor at $p_N = k_N$ is given by*

$$\begin{aligned}\mathcal{F}_N^l(p_1, \dots, p_N | k_0, \dots, k_N)(\psi_N, \phi_{N+1}) &\sim \frac{i}{k_N - p_N} \times \\ \left[(\mathcal{F}_{N-1}^l \otimes Id_{N,N})(\psi_N, \phi_{N+1}) - \sigma(Id_{1,0} \otimes \mathcal{F}_{N-1}^l)(\hat{S}^{1,N} \dots \hat{S}^{N-1,N} \psi_N, \hat{S}_{0,N} \dots \hat{S}_{N-1,N} \phi_{N+1}) \right] &.\end{aligned}\quad (3.20)$$

Here we abbreviated $\mathcal{F}_{N-1}^l = \mathcal{F}_{N-1}^l(p_1, \dots, p_{N-1} | k_0, \dots, k_{N-1})$. Poles at other $p_j = k_l$ can be obtained using the exchange property.

Proof. We follow the ideas of the proof of theorem 2. We consider those terms where p_N and k_N are coupled to x_N and there is a singularity. Similar to the scalar case, the only two possibilities are if x_N is larger, or smaller than any of the other coordinates. To be specific we first consider the following two cases:

$$x_1 < \dots < x_{N-1} < x_N, \quad 0 < x_N \quad \text{and} \quad x_N < x_1 < \dots < x_{N-1}, \quad x_N < 0.$$

In the first case (3.10) gives no action on the two vectors, therefore this term yields

$$\frac{i}{k_N - p_N} \mathcal{F}_{N-1}^l \otimes Id_{N,N}. \quad (3.21)$$

In the second case the wave function is obtained by symmetrization from (3.10):

$$\chi_{N+1}(0, x_1, \dots, x_N | \{k\}) = \sigma^N P_{N-1,N} \dots P_{01} \chi_{N+1}(x_N, 0, x_1, \dots, x_{N-1} | \{k\}).$$

Here we used

$$(P_{01} P_{12} \dots P_{N-1,N})^{-1} = P_{N-1,N} \dots P_{01}.$$

To obtain the coefficient of the exponential

$$e^{i(k_1 x_1 + \dots + k_N x_N)}$$

we have to consider the permutation $Q \in S_{N+1}$

$$Q(\{k_0, \dots, k_N\}) = \{k_N, k_0, \dots, k_{N-1}\}$$

For this permutation the corresponding linear operator is

$$\mathcal{Q}(Q, \{k\}) = \hat{S}_{0,N} \dots \hat{S}_{N-1,N}.$$

Performing similar steps for the dual vector we obtain the scalar product

$$\sigma \left\langle P_{N-1,N} \dots P_{1,2} \hat{S}^{1,N} \dots \hat{S}^{N-1,N} \phi_N | U_l^{(0)} P_{N-1,N} \dots P_{12} P_{0,1} \hat{S}_{0,N} \dots \hat{S}_{N-1,N} \phi_{N+1} \right\rangle_N.$$

This is equivalent to

$$\sigma \left\langle P_{12} \dots P_{N-1,N} \hat{S}^{1,N} \dots \hat{S}^{N-1,N} \phi_N | P_{12} \dots P_{N-1,N} U_l^{(1)} \hat{S}_{0,N} \dots \hat{S}_{N-1,N} \phi_{N+1} \right\rangle_N.$$

Moreover the scalar product is invariant with the permutation of vector spaces within $V^{(N)}$ therefore the above scalar product is equivalent to

$$\sigma \left\langle \hat{S}^{1,N} \dots \hat{S}^{N-1,N} \phi_N | U_l^{(1)} \hat{S}_{0,N} \dots \hat{S}_{N-1,N} \phi_{N+1} \right\rangle_N.$$

For the contribution in question the above scalar product is equivalent to the action of the operator

$$- \frac{i\sigma}{k_N - p_N} (Id_{1,0} \otimes \mathcal{F}_{N-1}^l) \left(\hat{S}^{1,N} \dots \hat{S}^{N-1,N} \phi_N, \hat{S}_{0,N} \dots \hat{S}_{N-1,N} \phi_{N+1} \right). \quad (3.22)$$

Adding the contributions (3.21)-(3.22), performing the summations over the remaining possibilities for the permutations, and using the Yang-Baxter equation one obtains finally the statement (3.20). \square

The residue equation (3.20) involves the parameter σ which distinguishes the bosonic and fermionic cases. It is useful to write down a relation which does not depend on σ . Therefore we introduce one more S -matrix, namely

$$\tilde{S}_{j,k} = \sigma \hat{S}_{j,k} \quad \tilde{S}^{j,k} = \sigma \hat{S}^{j,k}. \quad (3.23)$$

Then the kinematical pole equation reads

$$\begin{aligned} \mathcal{F}_N^l(p_1, \dots, p_N | k_0, \dots, k_N) (\psi_N, \phi_{N+1}) &\sim \frac{i}{k_N - p_N} \times \\ &\left[(\mathcal{F}_{N-1}^l \otimes Id_{N,N}) (\psi_N, \phi_{N+1}) - (Id_{1,0} \otimes \mathcal{F}_{N-1}^l) (\tilde{S}^{1,N} \dots \tilde{S}^{N-1,N} \psi_N, \tilde{S}_{0,N} \dots \tilde{S}_{N-1,N} \phi_{N+1}) \right]. \end{aligned} \quad (3.24)$$

It can be shown that an analogous equation (with the \tilde{S} operators involved) holds for the bilinear operators and arbitrary higher body local operators as well, irrespective of the statistics of the model. As a final remark we note that equation (3.24) can be considered as a non-relativistic version of the kinematical pole axiom known in the form factor bootstrap in integrable relativistic QFT's [36].

We conjecture that in the case of the field operator the recursion relation (3.20) together with the exchange properties (3.18) determine the form factors uniquely. Then, at least in principle they can be constructed with a procedure similar to the one presented in [70] for the case of the relativistic $O(3)$ σ -model. We leave this problem to further research.

In the case of the bilinear operators it is expected that the recursion relations are not restrictive enough. However, similar to the one-component case there is a useful asymptotic condition:

Theorem 4. *The asymptotic behaviour of the form factors of the bilinear operators is given by*

$$\lim_{p_1 \rightarrow \infty} \mathcal{F}_N^{jk}(p_1, \dots, p_N | k_1, \dots, k_N)(\psi_N, \phi_N) = \mathcal{F}_{N-1}^k(p_2, \dots, p_N | k_1, \dots, k_N)(U_j^{(1)} \psi_N, \phi_N). \quad (3.25)$$

Proof. The statement of the theorem is a direct consequence of the relation

$$\lim_{p_1 \rightarrow \infty} \Psi_j(0) \chi_N(0, x_1, \dots, x_{N-1} | p_1, \dots, p_N, \omega_N) \sim \chi_{N-1}(x_1, \dots, x_{N-1} | p_2, \dots, p_N, U_j^{(1)} \omega_N), \quad (3.26)$$

where similar to (2.20) the sign \sim indicates that on the l.h.s. only those terms are kept where p_1 is attached to $x_0 = 0$. Equation (3.26) is checked easily using the definitions (3.10)-(3.11) and the limiting values

$$\lim_{p_j \rightarrow \infty} Y_{jk}^{ab} = \rho_{ab}.$$

□

We conjecture that the form factors of the bilinear operators are determined uniquely by the exchange relations, the kinematical pole axiom (3.24) and the asymptotic condition (3.25). Also, we note that (3.25) can be considered as a non-relativistic version of the factorization property known in relativistic integrable QFT [71].

4 Two-component systems: the nested Bethe Ansatz in infinite volume

In this section we consider the two-component models. As a first step we construct the so-called nested Bethe Ansatz states. Our approach is somewhat different from the usual one: the results presented here apply directly in infinite volume, therefore we are not concerned with the periodicity of the wave functions. The connection to the finite volume states is made in section 6.

As a second step we also introduce the “magnonic form factors” which are matrix elements of local operators on the nested BA states. We investigate the analytic properties of these objects and obtain a set of “magnonic form factor equations”.

4.1 The nested BA states

We consider the infinite volume Bethe Ansatz states defined in (3.10) in the case of $V = \mathbb{C}^2$. The basis in V is formed by the two vectors $|+\rangle$ and $|-\rangle$. Our aim here is to specify the vectors ω_N entering the Bethe Ansatz states.

We consider an “auxiliary space” $V_a = \mathbb{C}^2$ and the operator (also called the “monodromy matrix”)

$$T(u|p) = X_{aN}(u - p_N) \dots X_{a1}(u - p_1). \quad (4.1)$$

The trace in auxiliary space is called the transfer matrix:

$$t(u|p) = \text{Tr}_a T(u|p). \quad (4.2)$$

The operator (4.1) can be viewed as the monodromy matrix of an inhomogeneous spin chain, where the rapidities p_j play the role of inhomogeneities. Then the standard Algebraic Bethe Ansatz techniques can be used to construct Bethe states in $V^{(N)}$.

To establish the notations we recall that the rational R -matrix is defined as

$$R(u, c) = \frac{1}{u + ic} \begin{pmatrix} u + ic & & & \\ & u & ic & \\ & ic & u & \\ & & & u + ic \end{pmatrix}. \quad (4.3)$$

Comparing to the formula (3.12) note that

$$X(u) = S_1(u, c)R(u, -\sigma c) \quad S_1(u, c) = \frac{\sigma u - ic}{u + ic}. \quad (4.4)$$

Here $S_1(u)$ is the “one-particle” S-matrix, which describes the amplitude associated to the exchange of two particles with the same spin. In the fermionic case it is equal to (-1) , whereas in the bosonic case it is just the Lieb-Liniger amplitude.

In order to conform with the conventions of the spin chain literature we define the normalized monodromy matrix

$$\tilde{T}(u|p) = R_{aN}(u - p_N) \dots R_{a1}(u - p_1). \quad (4.5)$$

Here it is understood that in the R -operators the coupling constant is $-\sigma c$. With respect to the auxiliary space it is written as

$$\tilde{T}(u|p) = \begin{pmatrix} \tilde{A}(u|p) & \tilde{B}(u|p) \\ \tilde{C}(u|p) & \tilde{D}(u|p) \end{pmatrix}.$$

The commutation relations of the elements of the monodromy matrix can be expressed in the compact form

$$R(u - v)\tilde{T}(u) \otimes \tilde{T}(v) = \tilde{T}(v) \otimes \tilde{T}(u)R(u - v), \quad (4.6)$$

which is understood as an equation in the tensor product of two auxiliary spaces. Eq. (4.6) follows from a repeated use of the Yang-Baxter equations (3.5). It follows from (4.6) that the transfer matrices $t(u|p)$ form a commuting set of operators.

We fix the reference state $|+\rangle_N = |++\dots+\rangle_N \in V^{(N)}$. The $B(\mu)$ -operators can be considered as creation operators of (interacting) spin-waves of $|-\rangle$ spins acting on the reference state. The parameter μ describes the rapidity of the spin wave and is often called the magnonic rapidity.

We define the function $\omega_N(\{p\}_N, \{\mu\}_M) : \mathbb{C}^N \times \mathbb{C}^M \rightarrow V^{(N)}$ as

$$\omega_N(\{p\}_N, \{\mu\}_M) = \tilde{B}(\mu_1 + i\sigma c/2|\{p\}) \dots \tilde{B}(\mu_M + i\sigma c/2|\{p\})|+\rangle_N. \quad (4.7)$$

The shift of $i\sigma c/2$ is introduced for technical reasons. According to (4.6) the B -operators commute with each other, therefore the function $\omega_N(\{p\}_N, \{\mu\}_M)$ is completely symmetric with respect to the set $\{\mu\}$. The exchange properties with respect to the set $\{p\}$ are determined once more by the Yang-Baxter relation:

Lemma 5. *The function $\omega_N(\{p\}_N, \{\mu\}_M)$ satisfies*

$$\omega_N(p_1, \dots, p_{j+1}, p_j, \dots, p_N, \{\mu\}) = \hat{R}_{j,j+1}\omega_N(p_1, \dots, p_j, p_{j+1}, \dots, p_N, \{\mu\}), \quad (4.8)$$

where $\hat{R}_{j,j+1} = P_{j,j+1}R_{j,j+1}(p_j - p_{j+1})$.

Proof. A modified form of the Yang-Baxter equation (3.5) is

$$\begin{aligned} Y_{j,j+1}(p_j - p_{j+1})X_{j+1,a}(u - p_{j+1})X_{ja}(u - p_j) = \\ X_{j+1,a}(u - p_j)X_{ja}(u - p_{j+1})Y_{j,j+1}(p_j - p_{j+1}). \end{aligned} \quad (4.9)$$

This implies

$$B(\mu|p_1, \dots, p_{j+1}, p_j, \dots, p_N)\hat{R}_{j,j+1} = \hat{R}_{j,j+1}B(\mu|p_1, \dots, p_j, p_{j+1}, \dots, p_N). \quad (4.10)$$

By commuting \hat{R} through the B -operators and using the fact that \hat{R} acts trivially on the reference state we obtain the statement (4.8). \square

This leads immediately to the following lemma:

Lemma 6. *The states defined by (4.7) satisfy the exchange property*

$$\mathcal{Q}(P, p) \omega_N(\{p\}_N, \{\mu\}_M) = \prod_{\substack{j < l \\ Pj > Pl}} S_1(p_j - p_l) \times \omega_N(\{Pp\}_N, \{\mu\}_M) \quad (4.11)$$

for arbitrary $P \in S_N$.

Proof. It is enough to check the statement for the elementary permutations. Then the statement follows from equations (4.8) and (4.4). \square

Finally we obtain the main statement about the nested Bethe Ansatz states:

Theorem 5. *The coordinate space wave functions defined as*

$$\chi_N(x|\{p\}_N, \{\mu\}_M) = \frac{1}{\sqrt{N!}} \sum_{P \in S_N} e^{i\langle Pp|x \rangle} \prod_{\substack{j < l \\ Pj > Pl}} S_1(p_j - p_l) \omega_N(\{Pp\}_N, \{\mu\}_M), \quad (4.12)$$

where $\omega_N(\{p\}_N, \{\mu\}_M)$ is given by (4.7), are eigenstates of the Hamiltonian (3.1). Here it is understood that (4.12) is defined in the fundamental domain $x_1 < \dots < x_N$ and it is extended to the other domains by symmetry.

Proof. The statement follows from Lemma 6 and the fact that the states (3.10) are eigenstates. \square

It is easy to see that the wave function (4.12) possesses the following exchange property with respect to the set $\{p\}$:

$$\begin{aligned} \chi_N(x|p_1, \dots, p_j, p_{j+1}, \dots, p_N, \{\mu\}_M) = \\ S_1(p_j - p_{j+1}) \chi_N(x|p_1, \dots, p_{j+1}, p_j, \dots, p_N, \{\mu\}_M). \end{aligned} \quad (4.13)$$

On the other hand, χ_N is completely symmetric with respect to the sets $\{\mu\}$.

It is useful to obtain explicit representations for the vectors $\omega_N(\{p\}_N, \{\mu\}_M)$. Here we just present the known results [47] using the notations of [23]:

$$\omega_N(\{p\}_N, \{\mu\}_M) = \sum_{a_1, \dots, a_M=1}^N A(a_1, \dots, a_M) \sigma_-^{a_1} \dots \sigma_-^{a_M} |+\rangle, \quad (4.14)$$

where

$$A(a_1, \dots, a_M) = \frac{1}{M!} \sum_{R \in S_M} \prod_{1 \leq k < l \leq M} \frac{(R\mu)_l - (R\mu)_k + i\sigma c \epsilon(a_l - a_k)}{(R\mu)_l - (R\mu)_k} \prod_{l=1}^M \mathcal{A}((R\mu)_l, a_l), \quad (4.15)$$

where $\epsilon(a)$ is the sign function and the propagator of the spin waves is given by

$$\mathcal{A}(u, a) = \frac{-i\sigma c}{u - p_a - i\sigma c/2} \prod_{b=1}^{a-1} \frac{u - p_b + i\sigma c/2}{u - p_b - i\sigma c/2}. \quad (4.16)$$

With this we have finished the explicit construction of the nested Bethe Ansatz states. Note that we did not address the completeness of states; in the present approach the magnonic rapidities are arbitrary parameters, they are not assumed to satisfy the Bethe equations.

It is useful to consider the $\mu \rightarrow \infty$ limit of the wave function. We obtain the following statement:

Lemma 7. *Sending a magnonic rapidity to infinity yields the action of the overall spin lowering operator:*

$$\lim_{\mu_1 \rightarrow \infty} \mu_1 \chi_N(x|\{p\}_N, \mu_1, \dots, \mu_M) = -i\sigma c S_- \chi_N(x|\{p\}_N, \mu_2, \dots, \mu_M). \quad (4.17)$$

Proof. It is known from the Algebraic Bethe Ansatz that

$$\lim_{\mu \rightarrow \infty} \mu \tilde{B}(\mu | \{p\}_N) = -i\sigma c S_- . \quad (4.18)$$

This is easily seen from the construction of the transfer matrix (4.1) or from the explicit expression (4.14). Applying (4.18) to (4.7) and finally to (4.12) we obtain the statement of the lemma. \square

4.2 Magnonic Form Factors

The magnonic Form Factors are defined as the matrix elements of local operators on the nested Bethe states. They are scalar functions of four sets of rapidities.

In the case of the field operator the form factor is defined as

$$\begin{aligned} \mathcal{F}_N^l(\{p\}_N, \{\nu\}_{M'}, \{k\}_{N+1}, \{\mu\}_M) &= \lim_{\varepsilon \rightarrow 0} \sqrt{N+1} \int_{-\infty}^{\infty} dx_1 \dots dx_N \prod_{j=1}^N f_\varepsilon(x_j) \\ &\times \left\langle \chi_N(x_1, \dots, x_N | \{p\}, \{\nu\}) \left| U_l^{(0)} \chi_{N+1}(0, x_1, \dots, x_N | \{k\}, \{\mu\}) \right\rangle_N . \end{aligned} \quad (4.19)$$

Due to spin conservation

$$M' = M + \frac{l-1}{2}$$

In the case of the bilinear operators the form factors are defined as

$$\begin{aligned} \mathcal{F}_N^{lm}(\{p\}_N, \{\nu\}_{M'}, \{k\}_N, \{\mu\}_M) &= \lim_{\varepsilon \rightarrow 0} N \int_{-\infty}^{\infty} dx_1 \dots dx_{N-1} \prod_{j=1}^{N-1} f_\varepsilon(x_j) \\ &\times \left\langle U_l^{(1)} \chi_N(0, x_1, \dots, x_{N-1} | \{p\}, \{\nu\}) \left| U_m^{(1)} \chi_N(0, x_1, \dots, x_{N-1} | \{k\}, \{\mu\}) \right\rangle_{N-1} . \end{aligned} \quad (4.20)$$

Spin conservation requires

$$M' = M + \frac{m-l}{2} .$$

The $U_l^{(j)}$ operators entering the formulas above are the projectors introduced in (3.16).

It is important to establish the analytic structure of the magnonic form factors.

It follows from the properties of the nested wave functions that the form factors are completely symmetric with respect to the magnonic rapidities. With respect to the particle rapidities they possess the exchange property (4.13) (and its complex conjugate).

Concerning the kinematical poles we substitute (4.7) into (3.20) and using Lemma 6 we obtain

$$\begin{aligned} \mathcal{F}_N^l(\{p\} | \{k\}) &\left(\omega_N(p_1, \dots, p_N, \{\nu\}), \omega_{N+1}(k_0, \dots, k_N, \{\mu\}) \right) \sim \frac{i}{k_N - p_N} \times \\ &\left[(\mathcal{F}_{N-1}^l \otimes Id_{N,N}) \left(\omega_N(p_1, \dots, p_N, \{\nu\}), \omega_{N+1}(k_0, \dots, k_N, \{\mu\}) \right) - \right. \\ &- \prod_{j=0}^{N-1} S_1(k_j - k_N) \prod_{j=1}^{N-1} S_1(p_N - p_j) \times \\ &\left. \times \sigma(Id_{1,0} \otimes \mathcal{F}_{N-1}^l) \left(\omega_N(p_N, p_1, \dots, p_{N-1}, \{\nu\}), \omega_{N+1}(k_N, k_0, \dots, k_{N-1}, \{\mu\}) \right) \right] . \end{aligned} \quad (4.21)$$

Note that in both terms the Id operator acts on those vector spaces to which the rapidities p_N and k_N are attached. Taking the scalar product with respect to these spaces leads to two possibilities: Either the corresponding components are $+$ or $-$. In both cases the resulting

amplitudes are evaluated easily using the explicit representation (4.14). We arrive at the following “inhomogeneous” form factor recursion equation:

$$\begin{aligned}
\mathcal{F}_N^l(\{p\}_{1..N}, \{\nu\}_{1..M'} | \{k\}_{0..N}, \{\mu\}_{1..M}) &\sim \frac{i}{k_N - p_N} \times \\
&\times \left[1 - \prod_{t=1}^M S_{\frac{1}{2}}(\mu_t - k_N) \prod_{t=1}^{M'} S_{\frac{1}{2}}(p_N - \nu_t) \prod_{j=0}^{N-1} \tilde{S}_1(k_{jN}) \prod_{k=1}^{N-1} \tilde{S}_1(p_{Nk}) \right] \times \\
&\times \mathcal{F}_{N-1}^l(\{p\}_{1..N-1}, \{\nu\}_{1..M'} | \{k\}_{0..N-1}, \{\mu\}_{1..M}) \\
&+ \sum_{s=1}^{M'} \sum_{r=1}^M \frac{c^2}{(\mu_r - k_N - i\sigma c/2)(\nu_s - p_N + i\sigma c/2)} \times \\
&\left[\prod_{j=0}^{N-1} S_{\frac{1}{2}}(\mu_r - k_j) \prod_{j=1}^{N-1} S_{\frac{1}{2}}(p_j - \nu_s) \prod_{\substack{t=1 \\ t \neq r}}^M W(\mu_t - \mu_r) \prod_{\substack{t=1 \\ t \neq s}}^{M'} W(\nu_s - \nu_t) - \right. \\
&- \left. \prod_{\substack{t=1 \\ t \neq r}}^M S_{\frac{1}{2}}(\mu_t - k_N) W(\mu_r - \mu_t) \prod_{\substack{t=1 \\ t \neq s}}^{M'} S_{\frac{1}{2}}(p_N - \nu_t) W(\nu_t - \nu_s) \prod_{j=0}^{N-1} \tilde{S}_1(k_{jN}) \prod_{k=1}^{N-1} \tilde{S}_1(p_{Nk}) \right] \\
&\times \mathcal{F}_{N-1}^l(\{p\}_{1..N-1}, \{\nu\}_{1..\hat{s}..M'} | \{k\}_{0..N-1}, \{\mu\}_{1..\hat{r}..M}).
\end{aligned} \tag{4.22}$$

Here we used

$$\tilde{S}_1(u) = \sigma S_1(u) = \frac{u - i\sigma c}{u + ic} \quad S_{\frac{1}{2}}(u) = \frac{u + i\sigma c/2}{u - i\sigma c/2} \quad W(u) = \frac{u + i\sigma c}{u},$$

and in the last line it is understood that the magnonic rapidities ν_s and μ_r are not substituted into the form factor.

Analogous relations can be written down for the bilinear operators as well. In those cases the ranges for the different products change according to the number of rapidities present.

The interpretation of the residue equation (4.22) is the following. The kinematical poles arise when two particle rapidities approach each other; in the multi-component case this leads to different contributions corresponding to the different spin orientations. In the cases when this component is $|+\rangle$ the remaining part of the wave function has the same number of $|-\rangle$ spins, and its explicit polarization is described by the same sets of magnonic rapidities. This corresponds to the second and third lines of (4.22). On the other hand, when the singular piece of the wave function carries a $|-\rangle$ spin, this is associated with one of the magnonic rapidities from both states. Therefore the remaining part of the wave functions yields form factors with one less number of $|-\rangle$ spins, ie. one less magnonic rapidity, just as in the form factors on the seventh line of (4.22).

The above recursive equations can be called “inhomogeneous” in the sense that they can not be solved by considering fixed sets of $\{\mu\}$ and $\{\nu\}$ as spectator variables: any recursion procedure with a given number of magnonic rapidities involves all the form factors with one less magnonic rapidity. This makes the solution of the system (4.22) more involved.

Note that the recursion relation (4.22) has a different structure than the singularity properties of the scalar products in the general $sl(3)$ symmetric model considered in [26]. This is due to the fact that we considered directly the form factors and not the scalar products, and that we used the explicit form of the coordinate wave functions, as opposed to the algebraic construction of [26]. Therefore we do not find singularities associated with coinciding magnonic rapidities, but the kinematical poles of the particle rapidities yield also the “inhomogeneous” terms.

The magnonic form factors have the structure

$$\mathcal{F}_N(\{p\}_N, \{\nu\}_{M'} | \{k\}_N, \{\mu\}_M) = c^{M'+M} \mathcal{P}_N(\{p\}_N, \{\nu\}_{M'} | \{k\}_N, \{\mu\}_M) \times \prod_{j>l} F_{\min}(k_j - k_l) \prod_{j>l} F_{\min}(p_l - p_j) \prod_{j,l} \frac{1}{k_j - p_l} \prod_{j,l} \frac{1}{\mu_j - k_l + ic/2} \prod_{j,l} \frac{1}{\nu_j - p_l - ic/2}. \quad (4.23)$$

Here $F_{\min}(k)$ is the so-called minimal two-particle form factor responsible for the exchange properties; it satisfies the relation

$$F_{\min}(u) = S_1(-u) F_{\min}(-u).$$

The solutions are given by

$$F_{\min}(u) = \begin{cases} u & \text{for fermions} \\ \frac{u}{u-ic} & \text{for bosons.} \end{cases} \quad (4.24)$$

\mathcal{P}_N is a polynomial which is symmetric with respect to all four sets of variables.

Lemma 8. *The maximal degree of \mathcal{P}_N in its variables depends on the operator in question and the statistics of the model and is given as follows:*

- *In the fermionic case:*
 - *Field operators: \mathcal{P}_N is of order M' in the p variables and of order $M - 1$ in the k variables.*
 - *Bilinear operators: \mathcal{P}_N is of order $M' + 1$ in the p variables and of order $M + 1$ in the k variables.*
- *In the bosonic case:*
 - *Field operators: \mathcal{P}_N is of order $N - 1 + M'$ in the p variables and of order $N - 1 + M$ in the k variables.*
 - *Bilinear operators: \mathcal{P}_N is of order $N + M'$ in the p variables and of order $N + M$ in the k variables.*

Proof. The total degree of the form factor in the particle rapidities can be established using the arguments given in Lemmas 2 and 3 and the explicit form of the wave function (4.12). Then the degree of \mathcal{P}_N can be read off from (4.23). The main difference between the degrees in the fermionic and bosonic cases is a result of the different structure of the minimal two-particle form factor. \square

It follows from the Lemma that in the fermionic case (4.22) is restrictive enough to fix the form factors completely.

On the other hand, in the bosonic case the degree of \mathcal{P}_N is typically higher than the number of conditions provided by (4.22). Further constraints can be found by sending one of the magnonic rapidities to infinity: Lemma (7) and the Wigner-Eckhart theorem provide additional relations between different form factors. However, at the present moment it is not clear whether (4.22) can be supplemented with other conditions which would fix the form factors. These questions are left for further research.

5 A few explicit solutions for the Form Factors

In this section we present solutions to the recursive equations (4.22). We only consider cases when there is at most one $|-\rangle$ spin in the two states; for certain operators this leads to homogeneous recursive equations which can be solved using generalizations of already known techniques. The solution of the full inhomogeneous equations would require new techniques and is outside the scope of the present paper.

As an independent check of our results we also evaluated the coordinate Bethe Ansatz expressions for a low number of particles using `Mathematica`. In the cases $N = 1, 2, 3$ we

performed the comparisons analytically, whereas in the cases $N = 4, 5$ we could only do numeric checks. In all cases we found complete agreement. This provides a strong justification for the results presented below, especially for the bilinear operators in the bosonic case, where the recursion relations do not fix the form factors completely.

5.1 Fermions

In the fermionic case we have $\tilde{S}_1 = 1$ leading to simple recursive equations. Form factors with only $|+\rangle$ spins (no magnonic rapidities) vanish identically, because the one-component fermionic model is a free theory (on the level of form factors this follows from $\tilde{S}_1 = 1$ leading to vanishing kinematic poles).

5.1.1 The matrix elements $\langle N, 0 | \Psi_- | N + 1, 1 \rangle$

Here we consider matrix elements of the $|-\rangle$ spin field operator between a completely polarized state and a state with only one $|-\rangle$ spin:

$$\mathcal{F}_N^-(p_1, \dots, p_N | k_0, \dots, k_N, \mu).$$

It follows from (4.22) that the pole at $p_N = k_N$ is given by

$$\begin{aligned} \mathcal{F}_N^-(p_1, \dots, p_N | k_0, \dots, k_N, \mu) &\sim \frac{i}{k_N - p_N} \times \\ &\left(1 - \frac{\mu - k_N - ic/2}{\mu - k_N + ic/2}\right) \mathcal{F}_{N-1}^-(p_1, \dots, p_{N-1} | k_0, \dots, k_{N-1}, \mu) \\ &= \frac{c}{(k_N - p_N)(k_N - \mu - ic/2)} \mathcal{F}_{N-1}^-(p_1, \dots, p_{N-1} | k_0, \dots, k_{N-1}, \mu). \end{aligned} \quad (5.1)$$

The starting point for the recursion is the formal value at $N = 0$:

$$\mathcal{F}_0^-(k_0, \mu) = \frac{-ic}{k_0 - \mu - ic/2}.$$

The solution to equation (5.1) is

$$\mathcal{F}_N^-(p_1, \dots, p_N | k_1, \dots, k_{N+1}, \mu) = -i \left(\prod_j \frac{c}{k_j - \mu - ic/2} \right) \frac{\prod_{i < j} k_{ji} \prod_{i > j} p_{ij}}{\prod_{j,l} (k_j - p_l)}.$$

5.1.2 The matrix elements $\langle N, 1 | \Psi_+ | N + 1, 1 \rangle$

If $N > 1$ then the “inhomogeneous” term vanishes and we obtain the recursion equations

$$\begin{aligned} \mathcal{F}_N^+(p_1, \dots, p_N, \nu | k_0, \dots, k_N, \mu) &\sim \frac{i}{k_N - p_N} \\ &\left(1 - \frac{\mu - k_N - ic/2}{\mu - k_N + ic/2} \frac{\nu - p_N + ic/2}{\nu - p_N - ic/2}\right) \mathcal{F}_{N-1}^+(p_1, \dots, p_{N-1}, \nu | k_0, \dots, k_{N-1}, \mu) = \\ &\frac{c(\mu - \nu)}{(k_N - p_N)(\mu - k_N + ic/2)(\nu - p_N - ic/2)} \mathcal{F}_{N-1}^+(p_1, \dots, p_{N-1}, \nu | k_0, \dots, k_{N-1}, \mu). \end{aligned} \quad (5.2)$$

At $N = 1$ the inhomogeneous term in (4.22) is the only contribution yielding for the pole at $k_1 \rightarrow p_1$:

$$\begin{aligned} \mathcal{F}_1^+(p_1, \nu | k_0, k_1, \mu) &\sim \frac{i}{k_1 - p_1} \frac{ic}{\mu - k_1 + ic/2} \frac{-ic}{\nu - p_1 - ic/2} [S_{\frac{1}{2}}(\mu - k_0) - 1] \mathcal{F}_0^+(k_0) \\ &= \frac{i}{k_1 - p_1} \frac{ic}{\mu - k_1 + ic/2} \frac{-ic}{\nu - p_1 - ic/2} \frac{-ic}{\mu - k_0 + ic/2}. \end{aligned}$$

There is an analogous relation for the pole at $p_1 \rightarrow k_0$. The solution is

$$\mathcal{F}_1^+(p_1, \nu | k_0, k_1, \mu) = -i \frac{-ic}{\nu - p_1 - ic/2} \frac{ic}{\mu - k_1 + ic/2} \frac{ic}{\mu - k_0 + ic/2} \frac{k_0 - k_1}{(k_1 - p_1)(k_0 - p_1)}.$$

The solution to the recursive equation then reads

$$\begin{aligned} \mathcal{F}_N^+(p_1, \dots, p_N, \nu | k_0, \dots, k_N, \mu) = \\ c^3 (c(\mu - \nu))^{N-1} \left(\prod_j \frac{1}{\mu - k_j + ic/2} \right) \left(\prod_j \frac{1}{\nu - p_j - ic/2} \right) \frac{\prod_{i < j} k_{ji} \prod_{i > j} p_{ij}}{\prod_{j,l} (k_j - p_l)}. \end{aligned}$$

5.1.3 The matrix elements $\langle N, 0 | \Psi_+^\dagger \Psi_- | N, 1 \rangle$

The recursive relation is analogous to (5.1):

$$\begin{aligned} \mathcal{F}_N^{+-}(p_1, \dots, p_N | k_1, \dots, k_N, \mu) \sim \frac{i}{k_N - p_N} \times \\ \frac{c}{(k_N - p_N)(k_N - \mu - ic/2)} \mathcal{F}_{N-1}^{+-}(p_1, \dots, p_{N-1} | k_1, \dots, k_{N-1}, \mu). \end{aligned} \quad (5.3)$$

The starting value is

$$\mathcal{F}_1^{+-}(p_1 | k_1, \mu) = \frac{ic}{\mu - k_1 + ic/2}.$$

The solution is

$$\mathcal{F}^{+-}(p_1, \dots, p_N | k_1, \dots, k_N, \mu) = -ic^N \left(\sum_j (k_j - p_j) \right) \prod_j \frac{1}{\mu - k_j + ic/2} \det \frac{1}{k_j - p_l}. \quad (5.4)$$

5.1.4 The matrix elements $\langle N, 1 | \Psi_-^\dagger \Psi_- | N, 1 \rangle$

The recursion equation reads

$$\begin{aligned} \mathcal{F}_N^{--}(p_1, \dots, p_N, \nu | k_1, \dots, k_N, \mu) \sim \frac{i}{k_N - p_N} \times \\ \frac{c(\mu - \nu)}{(k_N - p_N)(\mu - k_N + ic/2)(\nu - p_N - ic/2)} \mathcal{F}_{N-1}^{--}(p_1, \dots, p_{N-1}, \nu | k_1, \dots, k_{N-1}, \mu). \end{aligned} \quad (5.5)$$

The starting value is

$$\mathcal{F}_1^{--}(p_1, \nu | k_1, \mu) = \frac{c^2}{(\mu - k_N + ic/2)(\nu - p_N - ic/2)}.$$

The solution is

$$\begin{aligned} \mathcal{F}_N^{--}(p_1, \dots, p_N, \nu | k_1, \dots, k_N, \mu) = (c(\mu - \nu))^{N-1} \left(\sum_j (k_j - p_j) \right) \\ \times \frac{c^2}{\prod_j (\mu - k_j + ic/2)(\nu - p_j - ic/2)} \det \frac{1}{k_j - p_l}. \end{aligned} \quad (5.6)$$

5.1.5 The matrix elements $\langle N, 1 | \Psi_+^\dagger \Psi_+ | N, 1 \rangle$

Here the recursion relation is similar to the previous case:

$$\begin{aligned} \mathcal{F}_N^{++}(p_1, \dots, p_N, \nu | k_1, \dots, k_N, \mu) \sim \\ \frac{c(\mu - \nu)}{(k_N - p_N)(\mu - k_N + ic/2)(\nu - p_N - ic/2)} \mathcal{F}_{N-1}^{++}(p_1, \dots, p_{N-1}, \nu | k_1, \dots, k_{N-1}, \mu). \end{aligned} \quad (5.7)$$

However, this relation is valid only for $N > 2$. At $N = 2$ only the inhomogeneous term contributes and we have for example the pole at $p_2 \rightarrow k_2$:

$$\begin{aligned}\mathcal{F}_2^{++}(p_1, p_2, \nu | k_1, k_2, \mu) &\sim \\ &\sim \frac{i}{k_2 - p_2} \frac{ic}{\mu - k_2 + ic/2} \frac{-ic}{\nu - p_2 - ic/2} [S_{\frac{1}{2}}(\mu - k_1) S_{\frac{1}{2}}(p_1 - \nu) - 1] \mathcal{F}_1^{++}(p_1 | k_1) \\ &= \frac{i}{k_2 - p_2} \frac{ic}{\mu - k_2 + ic/2} \frac{-ic}{\nu - p_2 - ic/2} \frac{ic(\mu - \nu + p_1 - k_1)}{(\mu - k_1 + ic/2)(\nu - p_1 - ic/2)}.\end{aligned}$$

There are similar pole relations for the other residues. The solution is

$$\begin{aligned}\mathcal{F}_2^{++}(p_1, p_2, \nu | k_1, k_2, \mu) &= \\ &c^3 \frac{(k_1 + k_2 - p_1 - p_2)(\nu - \mu + k_1 + k_2 - p_1 - p_2)}{(\mu - k_1 + ic/2)(\mu - k_2 + ic/2)(\nu - p_1 - ic/2)(\nu - p_2 - ic/2)} \det \frac{1}{k_j - p_l}.\end{aligned}$$

The solution to the recursive equations is then

$$\begin{aligned}\mathcal{F}_N^{++}(p_1, \dots, p_N, \nu | k_1, \dots, k_N, \mu) &= c(c(\mu - \nu))^{N-2} \frac{c^2}{\prod_j (\mu - k_j + ic/2)(\nu - p_j - ic/2)} \\ &\times \left(\sum_j (k_j - p_j) \right) (\nu - \mu + \sum_j (k_j - p_j)) \det \frac{1}{k_j - p_l}.\end{aligned}\tag{5.8}$$

5.2 Bosons

In the bosonic case $\tilde{S}_1(u) = (u - ic)/(u + ic)$ and the recursive equations for the polarized states (no magnonic rapidities) produce the form factors of the Lieb-Liniger model. Therefore it is expected, that in those cases, where the recursive equations are homogeneous with fixed magnonic rapidities, the form factors can be obtained as generalizations of the Lieb-Liniger formulas.

There are three cases where this occurs, with one magnon at most in each of the states:

- Matrix elements of the down spin field operator: $\langle N, 0 | \Psi_- | N + 1, 1 \rangle$
- Matrix elements of the spin-flip operator: $\langle N, 0 | \Psi_+^\dagger \Psi_- | N, 1 \rangle$
- Matrix elements of the density of down spins: $\langle N, 1 | \Psi_-^\dagger \Psi_- | N, 1 \rangle$

The next matrix elements to consider would be the ones $\langle N, 1 | \Psi_+^\dagger \Psi_+ | N, 1 \rangle$. However, in this case there is an inhomogeneous term at each step of the recursion; it is given by a Lieb-Liniger density form factor. Therefore, already this case is outside the scope of the present methods and it requires new ideas.

5.2.1 The matrix elements $\langle N, 0 | \Psi_- | N + 1, 1 \rangle$

The residue property at $p_N \rightarrow k_{N+1}$ is

$$\begin{aligned}\mathcal{F}_N^-(p_1, \dots, p_N | k_1, \dots, k_{N+1}, \mu) &\sim \frac{i}{k_{N+1} - p_N} \times \\ &\times \left[1 - \frac{\mu - k_{N+1} + ic/2}{\mu - k_{N+1} - ic/2} \prod_{j=1}^N S(k_{j,N+1}) \prod_{k=1}^{N-1} S(p_{Nk}) \right] \mathcal{F}_{N-1}^-(p_1, \dots, p_{N-1} | k_1, \dots, k_N, \mu).\end{aligned}\tag{5.9}$$

The starting point for the recursion is the formal result

$$\mathcal{F}_0^-(k_0, \mu) = \frac{-ic}{\mu - k_0 - ic/2}.$$

It follows from Lemma 8 that the recursion with respect to the p variables completely fixes this form factor. We find the following solution:

$$\mathcal{F}_N^-(p_1, \dots, p_N | k_1, \dots, k_{N+1}, \mu) = \prod_{i>j} \frac{k_i - k_j + ic}{p_i - p_j + ic} \frac{-ic}{\prod_j (\mu - k_j - ic/2)} \det \mathbb{M}, \quad (5.10)$$

where \mathbb{M} is an $N \times N$ matrix with entries

$$\mathbb{M}_{jk} = M_{jk} - M_{N+1,k}$$

with

$$M_{jk} = t(p_k, k_j) h_{\frac{1}{2}}(\mu, k_j) \frac{\prod_{m=1}^N h(p_m, k_j)}{\prod_{m=1}^{N+1} h(k_m, k_j)} + t(k_j, p_k) h_{\frac{1}{2}}(k_j, \mu) \frac{\prod_{m=1}^N h(k_j, p_m)}{\prod_{m=1}^{N+1} h(k_j, k_m)} \quad (5.11)$$

and

$$h(u) = u + ic \quad h_{\frac{1}{2}}(u) = u + ic/2 \quad t(u) = \frac{-c}{u(u + ic)}.$$

The first example is

$$\mathcal{F}_1^-(p | k_0, k_1, \mu) = \frac{k_1 - k_0}{k_1 - k_0 - ic} \frac{ic^2(k_0 + k_1 - 2\mu)}{(\mu - k_0 - ic/2)(\mu - k_1 - ic/2)(p - k_0)(p - k_1)}.$$

An alternative expression for the same function is a generalization of the formula (2.23):

$$\mathcal{F}_N^-(\{p\}_N | \{k\}_{N+1}, \mu) = \frac{iP}{c^{N+1}} \prod_{j>l} \frac{1}{k_j - k_l - ic} \prod_{j>l} \frac{1}{p_j - p_l + ic} \frac{-ic}{\prod_j (\mu - k_j - ic/2)} \prod_{j,k} \frac{1}{k_j - p_l}. \quad (5.12)$$

Here

$$\begin{aligned} P = & \sum_{\alpha_j=0,1} \sum_{\beta_l=0,1} (-1)^{\sum_j \alpha_j + \sum_l \beta_l} \prod_{i1<i2} (p_{i1} - p_{i2} + (\alpha_{i1} - \alpha_{i2})ic) \times \\ & \times \prod_{i1<i2} (k_{i1} - k_{i2} + (\beta_{i1} - \beta_{i2})ic) \prod_{i1,i2} (p_{i1} - k_{i2} - (\alpha_{i1} - \beta_{i2})ic) \times \\ & \times \prod_j (\mu - k_j + (2\beta_j - 1)ic/2). \end{aligned} \quad (5.13)$$

It is easy to see that (5.12) satisfies all the conditions and the starting value for the recursion, therefore it is equivalent to (5.10).

5.2.2 The matrix elements $\langle N, 0 | \Psi_+^\dagger \Psi_- | N, 1 \rangle$

In this case the recursion relation is analogous to (5.9) and we found the following solution, which is given by a generalization of (2.21):

$$\begin{aligned} \mathcal{F}_N^{+-}(\{p\} | \{k\}, \mu) = & \frac{i}{c} (-1)^{N(N-1)/2} \prod_{j>l} \frac{1}{k_j - k_l - ic} \prod_{j>l} \frac{1}{p_j - p_l + ic} \\ & \times \prod_{o=1}^N \prod_{l=1}^N (k_o - p_l + ic) \times \det V \times \frac{-ic}{\prod_j (\mu - k_j - ic/2)}. \end{aligned} \quad (5.14)$$

Here V is an $(N+1) \times (N+1)$ matrix with entries

$$\begin{aligned}
V_{jl} &= (p_l - \mu + ic/2)\tilde{t}(k_j, p_l) + \\
&\quad + (p_l - \mu - ic/2)\tilde{t}(p_l, k_j) \prod_{o=1}^N \frac{(p_l - k_o + ic)(p_l - p_o - ic)}{(p_l - k_o - ic)(p_l - p_o + ic)}, \quad j, l = 1 \dots N \\
V_{N+1,j} &= \prod_{o=1}^N \frac{p_o - p_j + ic}{k_o - p_j + ic} \quad \text{and} \quad V_{j,N+1} = 1, \quad j = 1 \dots N \\
V_{N+1,N+1} &= 0
\end{aligned} \tag{5.15}$$

and

$$\tilde{t}(u) = \frac{-i}{u(u+ic)}$$

An alternative representation for the same function is given by

$$\begin{aligned}
\mathcal{F}_N^{+-}(\{p\}_N | \{k\}_N, \mu) &= \\
&= \frac{-i}{2c^N} \prod_{j>l} \frac{1}{k_j - k_l - ic} \prod_{j>l} \frac{1}{p_j - p_l + ic} \times P \times \frac{-ic}{\prod_j (\mu - k_j - ic/2)} \prod_{j,k} \frac{1}{k_j - p_l}
\end{aligned} \tag{5.16}$$

with

$$\begin{aligned}
P &= \sum_{\alpha_j=0,1} \sum_{\beta_l=0,1} (-1)^{\sum_j \alpha_j + \sum_l \beta_l} \prod_{i1<i2} (p_{i1} - p_{i2} + (\alpha_{i1} - \alpha_{i2})ic) \times \\
&\quad \times \prod_{i1<i2} (k_{i1} - k_{i2} + (\beta_{i1} - \beta_{i2})ic) \prod_{i1,i2} (p_{i1} - k_{i2} - (\alpha_{i1} - \beta_{i2})ic) \times \\
&\quad \times \prod_j (\mu - k_j + (2\beta_j - 1)ic/2) \times \sum_{j=1}^N ((-1)^{\alpha_j} + (-1)^{\beta_j})
\end{aligned} \tag{5.17}$$

The first two cases are given explicitly as

$$\begin{aligned}
\mathcal{F}_1^{+-}(p|k, \mu) &= \frac{-ic}{\mu - k - ic/2} \\
\mathcal{F}_2^{+-}(p_1, p_2 | k_1, k_2, \mu) &= (c^2 - 4k_1k_2 + 2(k_1 + k_2 - p_1 - p_2)\mu + (k_1 + k_2)(p_1 + p_2)) \times \\
&\quad \times \frac{-ic}{(\mu - k_1 - ic/2)(\mu - k_2 - ic/2)} \frac{k_2 - k_1}{k_2 - k_1 - ic} \frac{p_2 - p_1}{p_2 - p_1 + ic} \times \\
&\quad \times \frac{-c(k_1 + k_2 - p_1 - p_2)}{(k_1 - p_1)(k_1 - p_2)(k_2 - p_1)(k_2 - p_2)}.
\end{aligned}$$

In the present case the kinematic recursion relation itself is not sufficient to fix the form factor. We compared the two representations above to each other and to the coordinate Bethe Ansatz results. We performed the analytical check up to $N = 3$ and numerical checks for $N = 4, 5$ with **Mathematica**. Complete agreement was found, which provides a very strong justification for the general case of $N > 5$.

5.2.3 The matrix elements $\langle N, 1 | \Psi_-^\dagger \Psi_- | N, 1 \rangle$

The residue property at $p_N \rightarrow k_N$ is

$$\begin{aligned}
\mathcal{F}_N^{--}(p_1, \dots, p_N, \nu | k_1, \dots, k_N, \mu) &\sim \frac{i}{k_N - p_N} \times \\
&\left[1 - \frac{\mu - k_N + ic/2}{\mu - k_N - ic/2} \frac{\nu - p_N - ic/2}{\nu - p_N + ic/2} \prod_{j=1}^{N-1} S(k_{jN}) \prod_{k=1}^{N-1} S(p_{Nk}) \right] \times \\
&\mathcal{F}_{N-1}^{--}(p_1, \dots, p_{N-1}, \nu | k_1, \dots, k_{N-1}, \mu).
\end{aligned} \tag{5.18}$$

One solution is

$$\begin{aligned} \mathcal{F}_N^{--}(\{p\}, \nu | \{k\}, \mu) &= \frac{-i}{c} (-1)^{N(N+1)/2} \prod_{j>l} \frac{1}{k_j - k_l - ic} \prod_{j>l} \frac{1}{p_j - p_l + ic} \\ &\times \det V \times \prod_{o=1}^N \prod_{l=1}^N (k_o - p_l + ic) \times \frac{c^2}{\prod_j (\mu - k_j - ic/2)(\nu - p_j + ic/2)}. \end{aligned} \quad (5.19)$$

Here V is an $(N+1) \times (N+1)$ matrix with entries

$$\begin{aligned} V_{jl} &= (p_l - \mu + ic/2)(p_l - \nu - ic/2) \tilde{t}(k_j, p_l) + \\ &\quad (p_l - \mu - ic/2)(p_l - \nu + ic/2) \tilde{t}(p_l, k_j) \prod_{o=1}^N \frac{(p_l - k_o + ic)(p_l - p_o - ic)}{(p_l - k_o - ic)(p_l - p_o + ic)}, \quad j, l = 1 \dots N \\ V_{N+1, j} &= \prod_{o=1}^N \frac{p_o - p_j + ic}{k_o - p_j + ic} \quad \text{and} \quad V_{j, N+1} = 1, \quad j = 1 \dots N \\ V_{N+1, N+1} &= 0 \end{aligned} \quad (5.20)$$

and

$$\tilde{t}(u) = \frac{-i}{u(u+ic)}.$$

An alternative expression reads

$$\begin{aligned} \mathcal{F}_N^{--}(\{p\}_N, \nu | \{k\}_N, \mu) &= \\ &c^{-(N+1)} \frac{\sum_j (k_j - p_j)}{\nu - \mu + 2 \sum_j (k_j - p_j)} \prod_{j>l} \frac{1}{k_j - k_l - ic} \prod_{j>l} \frac{1}{p_j - p_l + ic} \\ &\times P \times \frac{c^2}{\prod_j (\mu - k_j - ic/2)(\nu - p_j + ic/2)} \prod_{j,k} \frac{1}{k_j - p_l}. \end{aligned} \quad (5.21)$$

Here

$$\begin{aligned} P &= \sum_{\alpha_j=0,1} \sum_{\beta_l=0,1} (-1)^{\sum_j \alpha_j + \sum_l \beta_l} \prod_{i1 < i2} (p_{i1} - p_{i2} + (\alpha_{i1} - \alpha_{i2})ic) \times \\ &\times \prod_{i1 < i2} (k_{i1} - k_{i2} + (\beta_{i1} - \beta_{i2})ic) \prod_{i1, i2} (p_{i1} - k_{i2} - (\alpha_{i1} - \beta_{i2})ic) \times \\ &\times \prod_j (k_j - \mu - (2\beta_j - 1)\frac{ic}{2})(p_j - \nu - (2\alpha_j - 1)\frac{ic}{2}) \end{aligned} \quad (5.22)$$

The first two cases are given explicitly as

$$\begin{aligned} \mathcal{F}_1^{--}(p, \nu | k, \mu) &= \frac{c^2}{(\mu - k - ic/2)(\nu - p + ic/2)} \\ \mathcal{F}_2^{--}(p_1, p_2, \nu | k_1, k_2, \mu) &= \frac{c(k_1 + k_2 - p_1 - p_2)(k_1 - k_2)(p_1 - p_2)}{(k_1 - p_1)(k_1 - p_2)(k_2 - p_1)(k_2 - p_2)} \times \\ &\quad \frac{c^2}{(\mu - k_1 - ic/2)(\mu - k_2 - ic/2)(\nu - p_1 + ic/2)(\nu - p_2 + ic/2)} \times \\ &\quad (c^2(\tilde{\mu} - \tilde{\nu}) - (k_1 - k_2)^2 \tilde{\nu} - 2(k_1 + k_2 - p_1 - p_2)\tilde{\mu}\tilde{\nu} + \tilde{\mu}(p_1 - p_2)^2). \end{aligned} \quad (5.23)$$

Here we used the auxiliary variables

$$\tilde{\mu} = \mu - \frac{k_1 + k_2}{2} \quad \tilde{\nu} = \nu - \frac{p_1 + p_2}{2}.$$

Once again we compared the two representations above to each other and to the coordinate Bethe Ansatz results up to $N = 5$ and found complete agreement.

6 The nested Bethe Ansatz in a finite volume

The previous sections were concerned with the infinite volume form factors of the multi-component systems. Here we consider the two-component case in a finite volume.

First we recall the conditions for the periodicity of the nested Bethe Ansatz wave function.

Theorem 6. *The wave function (4.12) is periodic in a finite volume L iff the rapidities $\{p\}_N$ and $\{\mu\}_M$ satisfy the following coupled set of equations:*

$$e^{ip_j L} = \prod_{\substack{l=1 \\ l \neq j}}^N \tilde{S}_1(p_l - p_j) \prod_{l=1}^M \frac{\mu_l - p_j + i\sigma c/2}{\mu_l - p_j - i\sigma c/2} \quad j = 1 \dots N \quad (6.1)$$

$$\prod_{j=1}^N \frac{\mu_l - p_j + ic/2}{\mu_l - p_j - ic/2} = \prod_{\substack{j=1 \\ j \neq l}}^M \frac{\mu_l - \mu_j + ic}{\mu_l - \mu_j - ic} \quad l = 1 \dots M. \quad (6.2)$$

Proof. For simplicity we only consider the periodicity with respect to the variable x_1 . By symmetry it is enough to consider the periodicity condition

$$\chi_N(-L/2, x_2, \dots, x_N | \{p\}_N, \{\mu\}_M) = \chi_N(x_2, \dots, x_N, L/2 | \{p\}_N, \{\mu\}_M)$$

in the domain

$$-L/2 < x_2 < \dots < x_N < L/2.$$

For simplicity we only consider the coefficient of the term proportional to

$$e^{i(x_1 p_1 + \dots + x_N p_N)}, \quad (6.3)$$

this will lead to the condition (6.1) with $j = 1$. At $x_1 = -L/2$ the coefficient is equal to $\omega_N(\{p\}, \{\mu\})$. On the other hand, at $x_1 = L/2$ the coefficient is

$$\rho(Q^{-1}) \mathcal{Q}(Q, p) \omega_N(\{p\}, \{\mu\}),$$

where Q is the permutation giving $Qp = \{p_2, \dots, p_N, p_1\}$. Thus we obtain the condition

$$\omega_N(\{p\}, \{\mu\}) = e^{ip_1 L} \rho(Q^{-1}) \mathcal{Q}(Q, p) \omega_N(\{p\}, \{\mu\}). \quad (6.4)$$

It follows from the definition (4.1) and from $X_{a,1}(0) = \sigma P_{a,1}$ that

$$T(p_1 | \{p\}) = \sigma P_{12} P_{23} \dots P_{N-1,N} P_{a,N} \hat{S}_{1,N} \hat{S}_{1,N-1} \dots \hat{S}_{1,2}. \quad (6.5)$$

Here $\hat{S}_{1,j} = Y_{1,j}^{j-1,j}$. Taking the trace with respect to the auxiliary space

$$t(p_1 | \{p\}) = \sigma^N \rho(Q^{-1}) \hat{S}_{1,N} \hat{S}_{1,N-1} \dots \hat{S}_{1,2} = \sigma^N \rho(Q^{-1}) \mathcal{Q}(Q, p). \quad (6.6)$$

Using (6.4) and (6.6) the periodicity condition is expressed as

$$\omega_N(\{p\}, \{\mu\}) = e^{ip_1 L} \sigma^N t(p_1 | \{p\}) \omega_N(\{p\}, \{\mu\}). \quad (6.7)$$

The eigenvalue equation

$$t(u | \{p\}) \omega_N(\{p\}, \{\mu\}) = \Lambda(u | \{p\}) \omega_N(\{p\}, \{\mu\}) \quad (6.8)$$

can be solved by the standard methods of algebraic Bethe Ansatz. It is known [14] that the above equation is satisfied whenever the magnonic rapidities are solutions to the inhomogeneous Bethe equation (6.2). Then the eigenvalues read

$$\Lambda(u | \{p\}) = \prod_{j=1}^N S_1(u - p_j) \left(\prod_{l=1}^M \frac{\mu_l - u - i\sigma c/2}{\mu_l - u + i\sigma c/2} + \prod_{j=1}^N \frac{u - p_j}{u - p_j - i\sigma c} \prod_{l=1}^M \frac{\mu_l - u + 3i\sigma c/2}{\mu_l - u + i\sigma c/2} \right). \quad (6.9)$$

Substituting $u = p_1$ results in

$$\Lambda(p_1|\{p\}) = \prod_{j=1}^N \tilde{S}_1(u - p_j) \prod_{l=1}^M \frac{\mu_l - u - i\sigma c/2}{\mu_l - u + i\sigma c/2}. \quad (6.10)$$

with $\tilde{S}_1(u) = \sigma S_1(u)$. This leads to (6.1) with $j = 1$. The other conditions with $j = 2, \dots, N$ follow from the symmetry properties of the wave function. It can be checked using the Yang-Baxter equation that these equations guarantee the periodicity for the coefficients of all exponentials and not only (6.3) considered here. \square

It is important to note that the states obtained by the finite solutions to the equations (6.1)-(6.2) do not span the full Hilbert space: it is known that the Algebraic Bethe Ansatz construction (4.7) only produces the states which are highest weight with respect to the overall $SU(2)$ symmetry [9]. In order to obtain the non-highest weight states one must act with spin lowering operator. This is known to be equivalent (in the proper normalization) to sending one magnonic rapidity to infinity. Supplied with these solutions the eqs. (6.1)-(6.2) are believed to yield a complete set of states.

6.1 Form factors in finite volume

We define the finite volume form factors in the same way as in the case of the Lieb-Liniger model. For example, in the case of the field operator the form factor is given by

$$\begin{aligned} \mathbb{F}_N^l(\{p\}_N, \{\nu\}_{M'}, \{k\}_{N+1}, \{\mu\}_M) &= \sqrt{N+1} \int_{-L/2}^{L/2} dx_1 \dots dx_N \\ &\times \left\langle \chi_N(x_1, \dots, x_N | \{p\}, \{\nu\}) \left| U_l^{(0)} \chi_{N+1}(0, x_1, \dots, x_N | \{k\}, \{\mu\}) \right\rangle_N. \end{aligned} \quad (6.11)$$

Spin conservation requires $M' = M + \frac{L-1}{2}$. Analogous definitions can be given for the form factors of the bilinear operators.

Theorem 7. *The form factors are the same in finite and infinite volume. In other words, if both sets $\{\{p\}, \{\nu\}\}$ and $\{\{k\}, \{\mu\}\}$ are solution to the nested Bethe equations and there are no coinciding particle rapidities ($p_j \neq k_l$), then*

$$\mathbb{F}_N^l(\{p\}_N, \{\nu\}_{M'}, \{k\}_{N+1}, \{\mu\}_M) = \mathcal{F}_N^l(\{p\}_N, \{\nu\}_{M'}, \{k\}_{N+1}, \{\mu\}_M). \quad (6.12)$$

An analogous relation holds for the matrix elements of the bilinear operators.

Proof. The theorem can be proven with the same arguments as Theorem 1. The key idea is that when the wave functions are periodic, the contributions of the Newton-Leibniz formula at $x = \pm L/2$ cancel each other and only the contributions at $x = \pm 0$ remain, which exactly coincide with those given by the infinite volume regularization of the oscillating integrals. \square

Finally we note that the normalized form factors are given by

$$\langle \{p\}_N, \{\nu\}_{M'} | \Psi_l | \{k\}_{N+1}, \{\mu\}_M \rangle = \frac{\mathcal{F}_N^l(\{p\}_N, \{\nu\}_{M'}, \{k\}_{N+1}, \{\mu\}_M)}{\sqrt{\mathcal{N}(\{p\}_N, \{\nu\}_{M'}) \mathcal{N}(\{k\}_{N+1}, \{\mu\}_M)}}, \quad (6.13)$$

and similarly for the bilinear operators. Here \mathcal{N} denotes the norm of the eigenstates and it reads [26, 27, 28]

$$\mathcal{N}(\{p\}_N, \{\mu\}_M) = c^M \det \mathcal{G} \prod_{1 \leq j < k \leq M} \left(1 + \frac{c^2}{(\mu_j - \mu_k)^2} \right), \quad (6.14)$$

where \mathcal{G} is an $(N+M) \times (N+M)$ matrix, also called the generalized Gaudin-determinant. It is the Jacobian associated to the coupled set of nested BA equations and it is given by

$$\mathcal{G} = \begin{pmatrix} \mathcal{G}_{pp} & \mathcal{G}_{p\mu} \\ \mathcal{G}_{\mu p} & \mathcal{G}_{\mu\mu} \end{pmatrix},$$

where the elements are

$$\begin{aligned}
(\mathcal{G}_{pp})_{jk} &= \delta_{jk} \left(L - \sigma \sum_l \frac{2c'}{(c')^2 + (p_j - \mu_l)^2} \right) + \\
&\quad + \frac{1 + \sigma}{2} \left(\delta_{jk} \sum_l \frac{2c}{c^2 + (p_j - p_l)^2} - \frac{2c}{c^2 + (p_j - p_k)^2} \right) \\
(\mathcal{G}_{p\mu})_{jk} &= (\mathcal{G}_{\mu p})_{kj} = \sigma \frac{2c'}{(c')^2 + (p_j - \mu_k)^2} \\
(\mathcal{G}_{\mu\mu})_{jk} &= \delta_{jk} \left(\sum_l \frac{2c'}{(c')^2 + (p_l - \mu_j)^2} - \sum_o \frac{2c}{c^2 + (\mu_o - \mu_j)^2} \right) + \frac{2c}{c^2 + (\mu_k - \mu_j)^2}
\end{aligned} \tag{6.15}$$

with $c' = c/2$.

7 Conclusions and Outlook

We investigated the form factors of local operators in the multi-component Quantum Non-Linear Schrödinger equation. The main results of the present work are the following:

1. Establishing the analytic properties of the (infinite volume) form factors in the general M -component case; in particular the kinematical pole equation (3.24). This can be regarded as a non-relativistic version of the well-known kinematical pole axiom from integrable relativistic QFT.
2. In the two-component case introducing the magnonic form factors and determining their analytic structure, in particular the kinematical pole equation (4.22).
3. The solution of the equation (4.22) in a number of simple cases, involving at most one magnonic rapidity per state. This was presented in section 5.
4. Making a connection to the finite volume form factors, established by the Theorems 1 and 7, which state that the un-normalized form factors are exactly the same in finite and infinite volume. The normalized finite volume matrix elements are given by expressions (2.9) and (6.13).

The most interesting question seems to be whether or not new solutions of the kinematical pole equations can be found. A possible direction is to consider integral representations for the co-vector valued form factors in the spirit of the so-called off-shell Bethe Ansatz [37]. However, it is not clear if such formulas can be found or if they would be useful for the calculation of correlation functions.

Another interesting question is to consider expectation values of local (or non-local) operators in a finite volume. All the techniques presented here apply in the case where there are no coinciding particle rapidities; this lies at the heart of identifying the finite volume and infinite volume matrix elements. On the other hand, it is known from the one-component case that the mean values can always be expressed with the properly regularized diagonal limits of the infinite volume form factors. This leads to integral representations for the mean values [61, 62]. A natural generalization is to consider this problem in the multi-component case; research in this direction is in progress.

It would be interesting to consider the singularity properties of form factors in the framework of the Algebraic Bethe Ansatz. To our best knowledge previous works only considered scalar products and norms of Bethe states; our kinematical pole equation (4.22) appears to be new. A very natural step would be to derive its “finite volume” version from ABA. This could help in clarifying the relation between the finite volume expectation values and the infinite volume form factors.

In the end of Section 4 we noted that in the bosonic case the kinematical pole equation (4.22) does not contain enough information to determine the form factors. It is an important open question whether additional constraints can be found, which would make the recursion

equations constraining enough. Also, it is an interesting question why does our formula (5.19) work (at least in the cases $N \leq 5$) for the density operator of the down spin particles. We did try other ways to generalize known formulas, and only this one did reproduce the results from coordinate Bethe Ansatz. This question is also left for further research.

As it was already mentioned in the introduction, the infinite volume Quantum Inverse Scattering Method (QISM) yields a representation for the field operator in terms of the Faddeev-Zamolodchikov operators [44]. From this result (the so-called quantum Rosales expansion) the form factors can be simply read off using only the Faddeev-Zamolodchikov algebra [46]. The quantum Rosales expansion is established also in the multi-component case [72, 73] and this provides us an alternative way to obtain the form factors. We checked in a number of simple cases (with low particle number) that the results thus obtained coincide with those presented in Section 5 [74]. However, the QISM does not seem to lead to compact and manageable formulas for the generic form factors with higher particle number. On the other hand, it would be interesting to consider the two-point functions in this framework: it might be possible to derive integral series for the finite temperature and finite density correlations along the lines of [46].

In the present work we only considered the non-relativistic multi-component continuum systems. However, it is expected that the ideas developed here also apply to other model solvable by the nested Bethe Ansatz. In particular it is expected that the annihilation pole equation (3.24) (or its “magnonic version” (4.22) and its appropriate generalizations to higher rank cases) hold in other $sl(N)$ related models, for example in the $SU(N)$ symmetric Heisenberg spin chains.

The identification of the finite and infinite volume form factors is an important ingredient of the present work. It is very natural to ask: is this result valid in integrable relativistic QFT? As it was mentioned in the introduction, in the realm of integrable QFT the infinite volume form factors are obtained using the so-called form factor bootstrap program, which has been established for theories with both diagonal and non-diagonal S-matrices. On the other hand, less is known about the finite volume matrix elements. In [75] it was shown that a relation like our (2.9) (concerning the 1-component model) holds in the massive relativistic theories with diagonal scattering. However, in the relativistic case the equality is not exact: there are finite size effects (decaying exponentially with the volume) due to virtual particle-antiparticle pairs “travelling around the world”. Concerning non-diagonal scattering theories, for example the sine-Gordon model, it is known that the finite size spectrum can be obtained (up to exponential corrections) with essentially the same nested Bethe Ansatz construction, as applied in this work [51]. The introduction of the “magnonic form factors” is very natural also in the relativistic case, and it is expected that a relation like (6.13) holds as well, once again up to the exponential corrections. As it was remarked above, an interesting open question (both from the coordinate Bethe Ansatz and the integrable QFT point of view) is the treatment of the matrix elements with coinciding particle rapidities, or in other words the finite volume evaluation of the “disconnected pieces” of the form factors. The resolution of this question is important for the evaluation of finite temperature correlations in massive integrable QFT [61, 76, 77, 78].

Acknowledgements

We are grateful to Jean-Sébastien Caux for useful discussions and for bringing the paper [49] to our attention. Also, we acknowledge his contribution to finding our final formulas for the form factors of the bilinear operators, expressed as single determinants (see the footnote on page 8).

We are thankful to Gábor Takács and Jean-Sébastien Caux for useful comments about the manuscript.

B. P. was supported by the VENI Grant 016.119.023 of the NWO.

M. K. acknowledges funding from The Welch Foundation, Grant No. C-1739.

References

- [1] E. H. Lieb and W. Liniger, “Exact Analysis of an Interacting Bose Gas. I. The General Solution and the Ground State,” *Phys. Rev.* **130** (1963) 1605.
- [2] E. H. Lieb, “Exact Analysis of an Interacting Bose Gas. II. The Excitation Spectrum,” *Phys. Rev.* **130** (1963) no. 4, 1616–1624.
- [3] H. Bethe, “Zur Theorie der Metalle,” *Zeitschrift für Physik* **A71** (1931) 205.
- [4] R. Orbach, “Linear Antiferromagnetic Chain with Anisotropic Coupling,” *Phys. Rev.* **112** (1958) no. 2, 309–316.
- [5] L. R. Walker, “Antiferromagnetic Linear Chain,” *Phys. Rev.* **116** (1959) no. 5, 1089–1090.
- [6] C. N. Yang and C. P. Yang, “One-Dimensional Chain of Anisotropic Spin-Spin Interactions. I. Proof of Bethe’s Hypothesis for Ground State in a Finite System,” *Phys. Rev.* **150** (1966) no. 1, 321–327.
- [7] L. D. Faddeev, E. K. Sklyanin, and L. A. Takhtajan, “The Quantum Inverse Problem Method. 1,” *Theor. Math. Phys.* **40** (1980) 688–706.
- [8] L. Faddeev, “Instructive history of the quantum inverse scattering method,” *Acta Applicandae Mathematicae* **39** (1995) 69–84.
- [9] L. D. Faddeev, “Algebraic Aspects of the Bethe Ansatz,” *Int. J. Mod. Phys. A* **10** (1995) 1845–1878, [arXiv:hep-th/9404013](#).
- [10] V. E. Korepin, “Calculation of norms of Bethe wave functions,” *Comm. Math. Phys.* **86** (1982) 391.
- [11] N. A. Slavnov, “Calculation of scalar products of wave functions and form factors in the framework of the algebraic Bethe ansatz,” *Theoretical and Mathematical Physics* **79** (1989) 502–508.
- [12] N. A. Slavnov, “Nonequal-time current correlation function in a one-dimensional Bose gas,” *Theoretical and Mathematical Physics* **82** (1990) 273–282.
- [13] T. Kojima, V. E. Korepin, and N. A. Slavnov, “Determinant Representation for Dynamical Correlation Functions of the Quantum Nonlinear Schrödinger Equation,” *Communications in Mathematical Physics* **188** (1997) 657–689.
- [14] V. Korepin, N. Bogoliubov, and A. Izergin, *Quantum inverse scattering method and correlation functions*. Cambridge University Press, 1993.
- [15] K. K. Kozłowski, “Large-distance and long-time asymptotic behavior of the reduced density matrix in the non-linear Schrödinger model,” [arXiv:1101.1626 \[math-ph\]](#).
- [16] F. A. Berezin, G. P. Pokhil, and V. M. Finkelberg *Vestnik Most Gos. Univ.* **1** (1964) 21.
- [17] J. B. McGuire, “Study of Exactly Soluble One Dimensional N-Body Problems,” *Journal of Mathematical Physics* **5** (1964) no. 5, 622.
- [18] E. Brezin and J. Zinn-Justin *CR Acad. Sci, Paris* **B263** (1966) 670.
- [19] C. N. Yang, “Some Exact Results for the Many-Body Problem in one Dimension with Repulsive Delta-Function Interaction,” *Phys. Rev. Lett.* **19** (1967) 1312–1315.
- [20] C. N. Yang, “S Matrix for the One-Dimensional N-Body Problem with Repulsive or Attractive δ -Function Interaction,” *Phys. Rev.* **168** (1968) 1920–1923.
- [21] B. Sutherland, “Further Results for the Many-Body Problem in One Dimension,” *Phys. Rev. Lett.* **20** (1968) 98–100.
- [22] S. Belliard and E. Ragoucy, “The nested Bethe ansatz for ‘all’ closed spin chains,” *Journal of Physics A: Mathematical and Theoretical* **41** (2008) no. 29, 295202.
- [23] F. H. L. Essler, H. Frahm, F. Göhmann, A. Klümper, and V. E. Korepin, *The One-Dimensional Hubbard Model*. Cambridge University Press, Cambridge, 2005.

- [24] P. P. Kulish, “Representation of the Zamolodchikov-Faddeev algebra,” *Journal of Mathematical Sciences* **24** (1984) 208–215.
- [25] P. P. Kulish and N. Y. Reshetikhin, “Diagonalisation of $GL(N)$ invariant transfer matrices and quantum N-wave system (Lee model),” *Journal of Physics A: Mathematical and General* **16** (1983) no. 16, L591.
- [26] N. Y. Reshetikhin, “Calculation of the norm of bethe vectors in models with $SU(3)$ -symmetry,” *Journal of Mathematical Sciences* **46** (1989) 1694.
- [27] G.-D. Pang, F.-C. Pu, and B.-H. Zhao, “Norms of Bethe wave functions for the nonlinear Schrödinger model of spin-1/2 particles,” *Journal of Mathematical Physics* **31** (1990) no. 10, 2497–2500.
- [28] F. Göhmann and V. E. Korepin, “The Hubbard chain: Lieb-Wu equations and norm of the eigenfunctions,” *Physics Letters A* **263** (1999) 293–298, [arXiv:cond-mat/9908114](#).
- [29] V. Tarasov and A. Varchenko, “Asymptotic Solutions to the Quantized Knizhnik-Zamolodchikov Equation and Bethe Vectors,” [arXiv:hep-th/9406060](#).
- [30] V. Tarasov and A. Varchenko, “Solutions to the Quantized Knizhnik-Zamolodchikov Equation and the Bethe Ansatz,” [arXiv:hep-th/9411181](#).
- [31] S. Belliard, S. Pakuliak, and E. Ragoucy, “Universal Bethe Ansatz and Scalar Products of Bethe Vectors,” *SIGMA* **6** (2010) 94, [arXiv:1012.1455 \[math-ph\]](#).
- [32] G. Mussardo, “Off critical statistical models: Factorized scattering theories and bootstrap program,” *Phys. Rept.* **218** (1992) 215–379.
- [33] A. B. Zamolodchikov and A. B. Zamolodchikov, “Factorized S-matrices in two dimensions as the exact solutions of certain relativistic quantum field models,” *Annals Phys.* **120** (1979) 253–291.
- [34] M. Karowski and P. Weisz, “Exact Form-Factors in (1+1)-Dimensional Field Theoretic Models with Soliton Behavior,” *Nucl. Phys.* **B139** (1978) 455.
- [35] B. Berg, M. Karowski, and P. Weisz, “Construction of Green’s functions from an exact S matrix,” *Phys. Rev. D* **19** (1979) 2477–2479.
- [36] F. A. Smirnov, “Form-factors in completely integrable models of quantum field theory,” *Adv. Ser. Math. Phys.* **14** (1992) 1–208.
- [37] H. Babujian, A. Fring, M. Karowski, and A. Zapletal, “Exact form factors in integrable quantum field theories: the sine-Gordon model,” *Nuclear Physics B* **538** (1999) 535–586, [arXiv:hep-th/9805185](#).
- [38] H. Babujian and M. Karowski, “Towards the Construction of Wightman Functions of Integrable Quantum Field Theories,” *International Journal of Modern Physics A* **19** (2004) 34–49, [arXiv:hep-th/0301088](#).
- [39] F. A. Smirnov, “Dynamical symmetries of massive integrable models, 1. Form-factor bootstrap equations as a special case of deformed Knizhnik- Zamolodchikov equations,” *Int. J. Mod. Phys.* **A71B** (1992) 813–837.
- [40] F. A. Smirnov, “Dynamical symmetries of massive integrable models. 2. Space of states of massive models as space of operators,” *Int. J. Mod. Phys.* **A7S1B** (1992) 839–858.
- [41] A. B. Zamolodchikov, “Two point correlation function in scaling Lee-Yang model,” *Nucl. Phys.* **B348** (1991) 619–641.
- [42] J. Balog and P. Weisz, “Structure functions of the 2d $O(n)$ non-linear sigma models,” *Nuclear Physics B* **709** (2005) 329–380, [arXiv:hep-th/0409095](#).
- [43] M. Kormos, G. Mussardo, and B. Pozsgay, “Bethe Ansatz Matrix Elements as Non-Relativistic Limits of Form Factors of Quantum Field Theory,” *J. Stat. Mech.* (2010) P05014, [arXiv:1002.3387 \[cond-mat.stat-mech\]](#).
- [44] D. B. Creamer, H. B. Thacker, and D. Wilkinson, “Gelfand-Levitan method for operator fields,” *Phys. Rev. D* **21** (1980) no. 6, 1523–1528.

- [45] D. B. Creamer, H. B. Thacker, and D. Wilkinson, “Some exact results for the two-point function of an integrable quantum field theory,” *Phys. Rev. D* **23** (1981) no. 12, 3081–3084.
- [46] D. B. Creamer, H. B. Thacker, and D. Wilkinson, “A study of correlation functions for the delta-function Bose gas,” *Physica D* **20** (1986) 155.
- [47] A. G. Izergin, V. E. Korepin, and N. Y. Reshetikhin, “Correlation functions in a one-dimensional Bose gas,” *J. Phys A* **20** (1987) 4799.
- [48] S. Pakuliak, “Annihilation Poles for Form Factors in the Xxz Model,” *International Journal of Modern Physics A* **9** (1994) 2087–2102, [arXiv:hep-th/9307090](#).
- [49] S. Palzer, C. Zipkes, C. Sias, and M. Köhl, “Quantum Transport through a Tonks-Girardeau Gas,” *Phys. Rev. Lett.* **103** (2009) 150601.
- [50] C. J. M. Mathy, M. B. Zvonarev, and E. Demler, “Quantum flutter of supersonic particles in one-dimensional quantum liquids,” [arXiv:1203.4819 \[cond-mat.quant-gas\]](#).
- [51] G. Fehér and G. Takács, “Sine-Gordon form factors in finite volume,” *Nuclear Physics B* **852** (2011) 441–467, [arXiv:1106.1901 \[hep-th\]](#).
- [52] G. Z. Fehér, T. Pálmai, and G. Takács, “Sine-Gordon multisoliton form factors in finite volume,” *Phys. Rev. D* **85** (2012) 085005, [arXiv:1112.6322 \[hep-th\]](#).
- [53] M. Gaudin, *La fonction d’onde de Bethe*. Paris: Masson, 1983.
- [54] G. J. Heckman and E. M. Opdam, “Yang’s System of Particles and Hecke Algebras,” *The Annals of Mathematics* **145** (1997) no. 1, 139–173. <http://www.jstor.org/stable/2951825>.
- [55] C. A. Tracy and H. Widom, “The dynamics of the one-dimensional delta-function Bose gas,” *Journal of Physics A Mathematical General* **41** (2008) 5204, [arXiv:0808.2491 \[math-ph\]](#).
- [56] C. N. Yang and C. P. Yang, “Thermodynamics of a One-Dimensional System of Bosons with Repulsive Delta-Function Interaction,” *J. Math. Phys.* **10** (1969) 1115.
- [57] T. C. Dorlas, “Orthogonality and completeness of the Bethe Ansatz eigenstates of the nonlinear Schroedinger model,” *Communications in Mathematical Physics* **154** (1993) 347–376.
- [58] M. Gaudin, “La fonction d’onde de Bethe pour les modèles exacts de la mécanique statistique,” *Commisariat à l’énergie atomique, Paris* (1983) .
- [59] A. G. Izergin and V. E. Korepin, “The quantum inverse scattering method approach to correlation functions,” *Comm. Math. Phys.* **94** (1984) 67.
- [60] V. E. Korepin, “Correlation functions of the one-dimensional Bose gas in the repulsive case,” *Comm. Math. Phys.* **94** (1984) 93.
- [61] B. Pozsgay and G. Takacs, “Form factors in finite volume II: disconnected terms and finite temperature correlators,” *Nucl. Phys.* **B788** (2008) 209–251, [arXiv:0706.3605 \[hep-th\]](#).
- [62] B. Pozsgay, “Mean values of local operators in highly excited Bethe states,” *J. Stat. Mech.* **2011** (2011) P01011, [arXiv:1009.4662 \[hep-th\]](#).
- [63] V. E. Korepin and N. A. Slavnov, “Form Factors in the Finite Volume,” *Int. J. Mod. Phys. B* **13** (1999) 2933–2941, [arXiv:math-ph/9812026](#).
- [64] B. Pozsgay, “Local correlations in the 1D Bose gas from a scaling limit of the XXZ chain,” *J. Stat. Mech.* **11** (2011) 17, [arXiv:1108.6224 \[cond-mat.stat-mech\]](#).
- [65] H. Babujian and M. Karowski, “Exact form factors in integrable quantum field theories: The sine-Gordon model. II,” *Nucl. Phys.* **B620** (2002) 407–455, [arXiv:hep-th/0105178](#).

- [66] H. Babujian and M. Karowski, “Sine-Gordon breather form factors and quantum field equations,” *J. Phys.* **A35** (2002) 9081–9104, [arXiv:hep-th/0204097](#).
- [67] M. Kormos, G. Mussardo, and A. Trombettoni, “Expectation Values in the Lieb-Liniger Bose Gas,” *Phys. Rev. Lett.* **103** (2009) 210404, [arXiv:0909.1336 \[cond-mat.stat-mech\]](#).
- [68] M. Kormos, G. Mussardo, and A. Trombettoni, “1D Lieb-Liniger Bose Gas as Non-Relativistic Limit of the Sinh-Gordon Model,” *Phys. Rev. A* **81** (2010) 043606, [arXiv:0912.3502 \[cond-mat.stat-mech\]](#).
- [69] E. Emsiz, E. Opdam, and J. Stokman, “Trigonometric Cherednik algebra at critical level and quantum many-body problems,” *Selecta Mathematica, New Series* **14** (2009) 571–605, [arXiv:0804.0046 \[math\]](#).
- [70] J. Balog and T. Hauer, “Polynomial form-factors in the O(3) nonlinear sigma model,” *Phys. Lett.* **B337** (1994) 115–121, [arXiv:hep-th/9406155](#).
- [71] G. Delfino, P. Simonetti, and J. L. Cardy, “Asymptotic factorisation of form factors in two- dimensional quantum field theory,” *Phys. Lett.* **B387** (1996) 327–333, [arXiv:hep-th/9607046](#).
- [72] F.-C. Pu and B.-H. Zhao, “Quantum Gelfand-Levitan equations for nonlinear Schrödinger model of spin-onehalf particles,” *Phys. Rev. D* **30** (1984) no. 10, 2253–2256.
- [73] F.-C. Pu, Y.-Z. Wu, and B.-H. Zhao, “Quantum inverse scattering method for multicomponent non-linear Schrodinger model of bosons or fermions with repulsive coupling,” *J. Phys. A* **20** (1987) no. 5, 1173.
- [74] B. Pozsgay and M. Kormos Unpublished.
- [75] B. Pozsgay and G. Takacs, “Form factors in finite volume I: form factor bootstrap and truncated conformal space,” *Nucl. Phys.* **B788** (2008) 167–208, [arXiv:0706.1445 \[hep-th\]](#).
- [76] F. H. L. Essler and R. M. Konik, “Finite-temperature dynamical correlations in massive integrable quantum field theories,” *J. Stat. Mech.* **0909** (2009) P09018, [arXiv:0907.0779 \[cond-mat.str-el\]](#).
- [77] B. Pozsgay and G. Takacs, “Form factor expansion for thermal correlators,” *J. Stat. Mech.* **11** (2010) 12, [arXiv:1008.3810 \[hep-th\]](#).
- [78] B. Doyon, “Finite-temperature form-factors: A Review,” *SIGMA* **3** (2007) 011, [arXiv:hep-th/0611066 \[hep-th\]](#).